Note on Undergraduate Real Analysis (K. Hunter, 2014)

Zepeng CHEN
The HK PolyU

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1 Sets and Functions

1.1 Sets

Definition 1.1 ($\mathcal{P}(X)$)

The power set $\mathcal{P}(X)$ of a set X is the set of all subsets of X.

Note on Example *If $X = \{1, 2, 3\}$, then*

$$\mathcal{P}(X) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{1, 2, 3\}\}.$$

Definition 1.2 (A^c)

A complement refers to the set of points not in A, i.e.,

$$A^c = \{x \in X : x \notin A\}.$$

Definition 1.3 ($A \backslash B$)

$$A - B = A \cap B^C$$

Theorem 1.1

- *Commutativity:*

$$A \cup B = B \cup A \quad A \cap B = B \cap A$$

- *Associativity:*

$$A \cup (B \cup C) = (A \cup B) \cup C \quad A \cap (B \cap C) = (A \cap B) \cap C$$

- *Distributive Laws:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- *De Morgan's laws:*

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

Definition 1.4 (Union and Intersection)

Let $\mathcal{C}$ be a collection of sets. Then the union of $\mathcal{C}$ and the intersection of $\mathcal{C}$ are

$$\bigcup \mathcal{C} = \{x : x \in X \text{ for some } X \in \mathcal{C}\} \quad \bigcap \mathcal{C} = \{x : x \in X \text{ for every } X \in \mathcal{C}\}.$$

Note on *The operations of union and intersection can be extended to infinite collections of sets as well. For example, let $S = (0, 1]$ and define $A_i = [(1/i), 1]$, then*

$$\begin{aligned}
\bigcup_{i=1}^{\infty} A_i &= \bigcup_{i=1}^{\infty} [(1/i), 1] = \{x \in (0, 1] : x \in [(1/i), 1] \text{ for some } i\} \\
&= \{x \in (0, 1]\} = (0, 1] \\
\bigcap_{i=1}^{\infty} A_i &= \bigcap_{i=1}^{\infty} [(1/i), 1] = \{x \in (0, 1] : x \in [(1/i), 1] \text{ for all } i\} \\
&= \{x \in (0, 1] : x \in [1, 1]\} = \{1\}
\end{aligned}$$

It is also possible to define unions and intersections over uncountable collections of sets. For example, we take $\Gamma = \{$ all positive real numbers $\}$ and $A_a = (0, a]$, then $\cup_{a \in \Gamma} A_a = (0, \infty)$ is an uncountable union.

Definition 1.5 (Cartesian product $X \times Y$)

The Cartesian product $X \times Y$ of sets X, Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

Definition 1.6 (Symmetric difference)

Symmetric difference of A and B refers to the set of points that are in one of the sets but not the other.

$$A \Delta B = (A - B) \cup (B - A)$$

Definition 1.7 ($A_i \uparrow$ and $A_i \downarrow$)

(1) $A_i \uparrow$ if $A_1 \subset A_2 \subset \dots$ and write $A_i \uparrow A$ if in addition $A = \cup_{i=1}^{\infty} A_i$.

(2) $A_i \downarrow$ if $A_1 \supset A_2 \supset \dots$ and write $A_i \downarrow A$ if in addition $A = \cap_{i=1}^{\infty} A_i$.

1.2 Function

Definition 1.8 (Support)

If $f : X \to \mathbb{R}$, the support of f is the closure of the set $\{x : f(x) \neq 0\}$.

Definition 1.9 (Graph)

The graph of a function $f : X \to Y$ is the subset G_f of $X \times Y$ defined by

$$G_f = \{(x, y) \in X \times Y : x \in X \text{ and } y = f(x)\}.$$

Definition 1.10 (Range or Image)

The range, or image, of a function $f : X \to Y$ *is the set of values*

$$\operatorname{ran} f = \{y \in Y : y = f(x) \text{ for some } x \in X\}.$$

Definition 1.11 (Onto function)

A function is onto if its range is all of Y*; that is, if*

$$\text{for every } y \in Y \text{ there exists } x \in X \text{ such that } y = f(x).$$

Definition 1.12 (One-to-one function)

A function is one-to-one if it maps distinct elements of X *to distinct elements of* Y*; that is, if*

$$x_1, x_2 \in X \text{ and } x_1 \neq x_2 \text{ implies that } f(x_1) \neq f(x_2).$$

Note on *An onto function is also called a surjection, a one-to-one function an injection, and a one-to-one, onto function a bijection.*

1.3 Indexed sets

Definition 1.13 (Indexed set)

A set X *is indexed by a set* I*, or* X *is an indexed set, if there is an onto function* $f : I \to X$*. We then write*

$$X = \{x_i : i \in I\}$$

where $x_i = f(i)$*.*

Definition 1.14 (Union and intersection of indexed collection)

If $\mathcal{C} = \{X_i : i \in I\}$ *is an indexed collection of sets* X_i*, then we denote the union and intersection of the sets in* $\mathcal{C}$ *by*

$$\bigcup_{i \in I} X_i = \{x : x \in X_i \text{ for some } i \in I\}, \quad \bigcap_{i \in I} X_i = \{x : x \in X_i \text{ for every } i \in I\}.$$

Proposition 1.1 (De Morgan)

If $\{X_i \subset X : i \in I\}$ *is a collection of subsets of a set* X*, then*

$$\left(\bigcup_{i \in I} X_i\right)^c = \bigcap_{i \in I} X_i^c, \quad \left(\bigcap_{i \in I} X_i\right)^c = \bigcup_{i \in I} X_i^c.$$

Theorem 1.2

Let $f : X \to Y$ *be a function. If* $\{Y_j \subset Y : j \in J\}$ *is a collection of subsets of* Y*, then*

$$f^{-1}\left(\bigcup_{j \in J} Y_j\right) = \bigcup_{j \in J} f^{-1}(Y_j), \quad f^{-1}\left(\bigcap_{j \in J} Y_j\right) = \bigcap_{j \in J} f^{-1}(Y_j);$$

and if $\{X_i \subset X : i \in I\}$ is a collection of subsets of X, then

$$f\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} f(X_i), \quad f\left(\bigcap_{i\in I} X_i\right) \subset \bigcap_{i\in I} f(X_i).$$

Note on Example *The only case in which we don't always have equality is for the image of an intersection. For example, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Let $A = (-1, 0)$ and $B = (0, 1)$. Then $f(A \cap B) = \varnothing$ and $f(A) \cap f(B) = (0, 1) \neq f(A \cap B)$.*

Definition 1.15 (Cartesian product of $\mathcal{C}$)

If $\mathcal{C} = \{X_i : i \in I\}$ is an indexed collection of sets X_i, the cartesian product of $\mathcal{C}$ is the set of functions that assign to each index $i \in I$ an element $x_i \in X_i$. That is,

$$\prod_{i\in I} X_i = \left\{ f : I \to \bigcup_{i\in I} X_i : f(i) \in X_i \text{ for every } i \in I \right\}.$$

Definition 1.16

Let

$$\Sigma = \{(s_1, s_2, s_3, \ldots, s_k, \ldots) : s_k = 0, 1\}$$

denote the set of all binary sequences; that is, sequences whose terms are either 0 or 1.

Note on Example *Let $2 = \{0, 1\}$. Then $\Sigma = 2^{\mathbb{N}}$.*

1.4 Relations

Definition 1.17 (Binary relation R)

A binary relation R on sets X and Y is a difinite relation between elements of X and elements of Y. We write xRy if $x \in X$ and $y \in Y$ are related.

Note on *A function $f : X \to Y$ determines a relation F on X and Y by: xFy iff $y = f(x)$.*

Note on *Two important types of relations are orders and equivalence relations.*

Definition 1.18 (Graph G_R)

The graph G_R of a relation R on X and Y is the subset of $X \times Y$ defined by

$$G_R = \{(x, y) \in X \times Y : xRy\}.$$

Definition 1.19 (Order $\preceq$)

A set X has a partial order "$\preceq$" if

(1) ***reflexivity.*** *$x \preceq x$ for all $x \in X$;*

(2) ***antisymmetry.*** *if $x \preceq y$ and $y \preceq x$, then $x = y$;*

(3) ***transitivity.*** *if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.*

Note on *We define a corresponding strict order $\prec$ by*

$$x \prec y \text{ if } x \preceq y \text{ and } x \neq y.$$

Definition 1.20 (Equivalence relation $\sim$)

A set X has an equivalence relationship "$\sim$" if

(1) ***reflexivity.*** $x \sim x$ *for all* $x \in X$;

(2) ***symmetry.*** *if* $x \sim y$, *then* $y \sim x$;

(3) ***transitivity.*** *if* $x \sim y$ *and* $y \sim z$, *then* $x \sim z$.

Note on Interpretation (Zhihu, 2018) *"$\sim$" is a kind of generalization of "$=$".*

Definition 1.21 ($X/\sim$)

For each $x \in X$, the set of elements equivalent to x,

$$[x/\sim] = \{y \in X : x \sim y\},$$

is called the equivalence class of x with respect to $\sim$. $[x/\sim]$ is also denoted simply by $[x]$. The set of equivalence classes of an equivalence relation $\sim$ on a set X is denoted by $X/\sim$.

Theorem 1.3

Let $\sim$ be an equivalence relation on a set X. Every equivalence class is non-empty, and X is the disjoint union of the equivalence classes of $\sim$.

Note on *If $x \in X$, then the symmetry of $\sim$ implies that $x \in [x]$.*

Definition 1.22 ($x \vee y$ and $x \wedge y$)

$$x \vee y = \max(x, y) \quad x \wedge y = \min(x, y)$$

1.5 Countable and Uncountable Sets

Definition 1.23 ($X \approx Y$ and $X \lesssim Y$)

(1) *Two sets X, Y have equal cardinality, written $X \approx Y$, if there is a one-to-one, onto map $f : X \to Y$.*

(2) *The cardinality of X is less than or equal to the cardinality of Y, written $X \lesssim Y$, if there is a one-to-one (but not necessarily onto) map $g : X \to Y$.*

Theorem 1.4 (Schröder-Bernstein)

If X, Y are sets such that there are one-to-one maps $f : X \to Y$ and $g : Y \to X$, then there is a one-to-one, onto map $h : X \to Y$.

Definition 1.24 (Finite / Countably infinite / Infinite / Countable / Uncountable)

A set X is:

(1) Finite if it is the empty set or $X \approx \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$;

(2) Countably infinite (or denumerable) if $X \approx \mathbb{N}$;

(3) Infinite if it is not finite;

(4) Countable if it is finite or countably infinite;

(5) Uncountable if it is not countable.

Note on Example *The set of squares*

$$S = \{1, 4, 9, 16, \ldots, n^2, \ldots\}$$

is countably infinite since $f : \mathbb{N} \to S$ defined by $f(n) = n^2$ is one-to-one and onto.

Definition 1.25

A set has the cardinality of the continuum if $X \approx \mathbb{R}$.

Proposition 1.2

A non-empty set X is countable iff there is an onto map $f : \mathbb{N} \to X$.

Proposition 1.3

The Cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite.

Theorem 1.5

A countable union of countable sets is countable.

Theorem 1.6

The power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ is uncountable.

2 Numbers

Proposition 2.1

The set of integers $\mathbb{Z}$ is countably infinite.

Definition 2.1 ($\mathbb{Q}$)

The set of rational numbers is defined by

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\},$$

where we may cancel common factors from the numerator and denominator, meaning that

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \quad \text{if and only if } p_1 q_2 = p_2 q_1.$$

Note on *We can construct $\mathbb{Q}$ from $\mathbb{Z}$ as the collection of equivalence classes in $\mathbb{Z} \times \mathbb{Z} \backslash \{0\}$ with respect to the equivalence relation $(p_1, q_1) \sim (p_2, q_2)$ if $p_1 q_2 = p_2 q_1$.*

Note on *The rational numbers are linearly ordered by their standard order. Moreover, this order is dense, meaning that if $r_1, r_2 \in \mathbb{Q}$ and $r_1 < r_2$, then there exists a rational number $r \in \mathbb{Q}$ between them with $r_1 < r < r_2$. For example, we can take*

$$r = \frac{1}{2}(r_1 + r_2)$$

. In addition, there are a lot of "gaps" between rational numbers which are filled by the irrational real numbers.

Theorem 2.1

The rational numbers are countably infinite.

Note on *Altenatively, we can write*

$$\mathbb{Q} = \bigcup_{q \in \mathbb{N}} \{p/q : p \in \mathbb{Z}\}$$

as a countable union of countable sets.

Definition 2.2 (sup **and** inf)

Suppose that $A \subset \mathbb{R}$ is a set of real numbers. If $M \in \mathbb{R}$ is an upper bound of A such that $M \leq M'$ for every upper bound M' of A, then M is called the least upper bound or supremum of A, denoted

$$M = \sup A.$$

If $m \in \mathbb{R}$ is a lower bound of A such that $m \geq m'$ for every lower bound m' of A, then m is called the greatest lower bound or infimum of A, denoted

$$m = \inf A.$$

If $A = \{x_i : i \in I\}$ is an indexed subset of $\mathbb{R}$, we also write

$$\sup A = \sup_{i \in I} x_i, \quad \inf A = \inf_{i \in I} x_i.$$

Definition 2.3 ($\overline{\mathbb{R}}$)

To specify the supremum and infimum, we introduce a system of extended real numbers

$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

If a set $A \subset \mathbb{R}$ is not bounded from above, then $\sup A = \infty$, and if A is not bounded from below, then $\inf A = -\infty$.

Theorem 2.2

If $x \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $x < n$.

Theorem 2.3

The set of real numbers is uncountable.

Proposition 2.2

If $A \subset \mathbb{R}$, then $M = \sup A$ iff:

(a) M is an upper boudn of A;

(b) for every $M' < M$ there exists $x \in A$ such that $x > M'$.

Note on

(1) If M is an upper bound of A, then $\sup A \leq M$;

(2) If A is nonempty and bounded from above, then for every $\epsilon > 0$, there exists $x \in A$ such that $x > \sup A - \epsilon$.

Proposition 2.3

Suppose that A, B are subsets of $\mathbb{R}$ such that $A \subset B$. Then $\sup A \leq \sup B$, and $\inf A \geq \inf B$.

Proposition 2.4

Suppose that A, B are nonempty sets of real numbers such that $x \leq y$ for all $x \in A$ and $y \in B$. Then $\sup A \leq \inf B$.

Proposition 2.5

If $c \geq 0$, then

$$\sup cA = c \sup A, \quad \inf cA = c \inf A.$$

If $c < 0$, then

$$\sup cA = c \inf A, \quad \inf cA = c \sup A.$$

Proposition 2.6

If A, B are nonempty sets, then

$$\sup(A+B) = \sup A + \sup B, \quad \inf(A+B) = \inf A + \inf B,$$

$$\sup(A-B) = \sup A - \inf B, \quad \inf(A-B) = \inf A - \sup B.$$

Proposition 2.7

Suppose that $\{x_{ij} : i \in I, j \in J\}$ is a doubly-indexed set of real numbers. Then

$$\sup_{(i,j) \in I \times J} x_{ij} = \sup_{i \in I} \left(\sup_{j \in J} x_{ij} \right).$$

3 Sequences

3.1 Convergence and limits

Proposition 3.1

A set $A \subset \mathbb{R}$ is bounded iff there exists a real number $M \geq 0$ such that

$$|x| \leq M \text{ for every } x \in A.$$

Definition 3.1

Let $A \subset \mathbb{R}$. The diameter of A is

$$\operatorname{diam} A = \sup\{|x - y| : x, y \in A\}.$$

Note on *Then a set is bounded iff its diameter is finite.*

Definition 3.2 (Sequence)

A sequence (x_n) of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$, where $x_n = f(n)$.

Note on Example *Fibonacci sequence (F_n): The nth term is given by*

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right], \quad \phi = \frac{1 + \sqrt{5}}{2}.$$

Definition 3.3 (Sequence converge)

A sequence (x_n) of real numbers converges to a limit $x \in \mathbb{R}$, written

$$x = \lim_{n \to \infty} x_n, \quad \text{or} \quad x_n \to x \text{ as } n \to \infty,$$

if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \quad \text{for all } n > N.$$

A sequence converges if it converges to some limit $x \in \mathbb{R}$, otherwise it diverges.

Proposition 3.2

If a sequence converges, then its limit is unique.

Definition 3.4 ($\lim_{n \to \infty} x_n = \infty$ **and** $\lim_{n \to \infty} x_n = -\infty$)

If (x_n) is a sequence then

$$\lim_{n \to \infty} x_n = \infty,$$

or $x_n \to \infty$ as $n \to \infty$, if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that

$$x_n > M \quad \text{for all } n > N.$$

Definition 3.5

(1) A sequence (x_n) of real numbers is bounded from above if there exists $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$.

(2) A sequence (x_n) of real numbers is bounded from below if there exists $m \in \mathbb{R}$ such that $x_n \geq m$ for all $n \in \mathbb{N}$.

(3) A sequence is bounded if it is bounded from above and below, otherwise it is unbounded.

Definition 3.6

A convergent sequence is bounded.

Note on *Boundedness is a necessary condition for convergence, and every unbounded sequence diverges. On the other hand, boundedness is not a sufficient condition for convergence.*

Definition 3.7

Let $P(x)$ denote a property of real numbers $x \in \mathbb{R}$. If (x_n) is a real sequence, then $P(x_n)$ holds eventually if there exists $N \in \mathbb{N}$ such that $P(x_n)$ holds for all $n > N$; and $P(x_n)$ holds infinitely often if for every $N \in \mathbb{N}$ there exists $n > N$ such that $P(x_n)$ holds.

Note on Example *Monotonicity, Linearity,*

Theorem 3.1 (Monotonicity)

If (x_n) and (y_n) are convergent sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Theorem 3.2 (Sandwich)

Suppose that (x_n) and (y_n) are convergent sequences of real numbers with the same limit L. If (z_n) is a sequence such that

$$x_n \leq z_n \leq y_n \quad \text{for all } n \in \mathbb{N},$$

then (z_n) also converges to L.

Corollary 3.1

If $x_n \to x$ as $n \to \infty$, then $|x_n| \to |x|$ as $n \to \infty$.

Theorem 3.3 (Linearity)

Suppose that (x_n) and (y_n) are convergent real sequences and $c \in \mathbb{R}$. Then the sequences

(cx_n), $(x_n + y_n)$, and $(x_n y_n)$ converge, and

$$\lim_{n\to\infty} cx_n = c \lim_{n\to\infty} x_n,$$
$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$
$$\lim_{n\to\infty} (x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

Definition 3.8 (Increasing / Decreasing / Monotone)

(1) A sequence (x_n) of real numbers is (strictly) increasing if $x_{n+1} \geq x_n$ ($x_{n+1} > x_n$) for all $n \in \mathbb{N}$.

(2) A sequence (x_n) of real numbers is decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

(3) A sequence (x_n) of real numbers is monotone if it is increasing or decreasing.

Theorem 3.4

A monotone sequence of real numbers converges iff it is bounded.

(1) If (x_n) is monotone increasing and bounded, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

We use $x_n \uparrow x$ indicate that (x_n) is a monotone increasing sequence that converges to x.

(2) If (x_n) is monotone decreasing and bounded, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

We use $x_n \downarrow x$ indicate that (x_n) is a monotone decreasing sequence that converges to x.

Note on Examples

(1) The geometric sequence (a^n) is strictly monotone decreasing if $0 < a < 1$, with $\lim_{n\to\infty} a^n = 0$, and strictly monotone increasing if $1 < a < \infty$, with $\lim_{n\to\infty} a^n = \infty$.

(2) The sequence $(x_n = \left(1 + \frac{1}{n}\right)^n)$ is strictly monotone increasing and converges to a limit $2 < e < 3$.

3.2 The $\limsup$ and $\liminf$

Definition 3.9 ($\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$)

Suppose that (x_n) is a sequence of real numbers. Then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n, \quad y_n = \sup\{x_k : k \geq n\},$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} z_n, \quad z_n = \inf\{x_k : k \geq n\}.$$

We also write

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} \left(\sup_{k\geq n} x_k \right), \quad \liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} x_k \right).$$

Note on Example

$$a_n = \begin{cases} 1, n \text{ is even} \\ -1/n, n \text{ is odd} \end{cases} \tag{1}$$

$\limsup_{n\to\infty} a_n = 1$ *and* $\liminf_{n\to\infty} a_n = 0$.

Definition 3.10

Suppose that (x_n) *is a sequence of real numbers and the sequences* (y_n), (z_n) *of possibly extended real numbers are given by Definition 3.9. Then*

$$\begin{aligned} \limsup_{n\to\infty} x_n &= \infty && \text{if } y_n = \infty \text{ for every } n \in \mathbb{N}, \\ \limsup_{n\to\infty} x_n &= -\infty && \text{if } y_n \downarrow -\infty \text{ as } n \to \infty, \\ \liminf_{n\to\infty} x_n &= -\infty && \text{if } z_n = -\infty \text{ for every } n \in \mathbb{N}, \\ \liminf_{n\to\infty} x_n &= \infty && \text{if } z_n \uparrow \infty \text{ as } n \to \infty. \end{aligned}$$

Note on Examples *Consider* $(x_n = 1 - \frac{1}{n})$, *we have*

$$y_n = \sup\left\{1 - \frac{1}{k} : k \geq n\right\} = 1, \quad z_n = \inf\left\{1 - \frac{1}{k} : k \geq n\right\} = 1 - \frac{1}{n}.$$

Thus,

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = 1.$$

Consider $(x_n = (-1)^{n+1}\left(1 + \frac{1}{n}\right))$, *we have*

$$y_n = \sup\{x_k : k \geq n\} = \begin{cases} 1 + 1/n & \text{if } n \text{ is odd,} \\ 1 + 1/(n+1) & \text{if } n \text{ is even,} \end{cases}$$

$$z_n = \inf\{x_k : k \geq n\} = \begin{cases} -[1 + 1/(n+1)] & \text{if } n \text{ is odd,} \\ -[1 + 1/n] & \text{if } n \text{ is even.} \end{cases}$$

And it follows that

$$\limsup_{n\to\infty} x_n = 1, \quad \liminf_{n\to\infty} x_n = -1.$$

Theorem 3.5

Let (x_n) *be a real sequence. Then*

$$y = \limsup_{n\to\infty} x_n$$

iff $-\infty \leq y \leq \infty$ *satisfies one of the following conditions.*

(1) $-\infty < y < \infty$ *and for every* $\epsilon > 0$*:*

(a) there exists $N \in \mathbb{N}$ *such that* $x_n < y + \epsilon$ *for all* $n > N$*;*

(b) *for every* $N \in \mathbb{N}$ *there exists* $n > N$ *such that* $x_n > y - \epsilon$.

(2) $y = \infty$ *and for every* $M \in \mathbb{R}$, *there exists* $n \in \mathbb{N}$ *such that* $x_n > M$, *i.e.*, (x_n) *is not bounded from above.*

(3) $y = -\infty$ *and for every* $m \in \mathbb{R}$ *there exists* $N \in \mathbb{N}$ *such that* $x_n < m$ *for all* $n > N$, *i.e.*, $x_n \to -\infty$ *as* $n \to \infty$.

Theorem 3.6

A sequence (x_n) *of real numbers converges iff*

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$$

are finite and equal, in which case

$$\lim_{n\to\infty} x_n = x.$$

Note on *Every sequence has a finite or infinite* $\limsup$, *but not every sequence has a limit.*

Note on Example *Let* $x_n = 1 + (-1)^n$. *Then* (x_n) *oscillates between* 0 *and* 2, *and*

$$\liminf_{n\to\infty} x_n = 0, \quad \limsup_{n\to\infty} x_n = 2.$$

The sequence is non-negative and its $\liminf$ *is* 0, *but the sequence does not converge.*

Corollary 3.2

Let (x_n) *be a sequence of real numbers. Then* (x_n) *converges with* $\lim_{n\to\infty} x_n = x$ *iff* $\limsup_{n\to\infty} |x_n - x| = 0$.

3.3 Cauchy sequences

Definition 3.11 (Cauchy sequence)

A sequence (x_n) *of real numbers is a Cauchy sequence if for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$|x_m - x_n| < \epsilon \quad \text{for all } m, n > N.$$

Theorem 3.7

A sequence of real numbers converges iff it is a Cauchy sequence.

3.4 Subsequences

Definition 3.12 (Subsequence)

A subsequence of a sequence (x_n) *is a sequence* (x_{n_k}) *of the form*

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots$$

where $n_1 < n_2 < n_3 < \cdots < n_k < \ldots$.

Definition 3.13

Let (x_n) be a sequence, where $x_n = f(n)$ and $f : \mathbb{N} \to \mathbb{R}$. A sequence (y_k), where $y_k = g(k)$ and $g : \mathbb{N} \to \mathbb{R}$, is a subsequence of (x_n) if there is a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $g = f \circ \phi$. In that case, we write $\phi(k) = n_k$ and $y_k = x_{n_k}$.

Proposition 3.3

Every subsequence of a convergent sequence converges to the limit of the sequence.

Corollary 3.3

If a sequence has subsequences that converge to different limits, then the sequence diverges.

Note on *The sequence $\left((-1)^{n+1}\right)$ has subsequences (1) and (-1) that converge to different limits, so it diverges.*

Definition 3.14

The limit set of a sequence (x_n) is the set

$$\{x \in \mathbb{R} : \text{ there is a subsequence } (x_{n_k}) \text{ such that } x_{n_k} \to x \text{ as } k \to \infty\}$$

of limits of all of its convergent subsequences.

Note on *The limit set of the divergent sequence $\left((-1)^{n+1}\right)$ contains two points, and is $\{-1, 1\}$.*

Theorem 3.8

Suppose that (x_n) is sequence of real numbers with limit set S. Then

$$\limsup_{n\to\infty} x_n = \sup S, \quad \liminf_{n\to\infty} x_n = \inf S.$$

3.5 The Bolzano-Weierstrass theorem

Theorem 3.9 (Bolzano-Weierstrass)

Every bounded sequence of real numbers has a convergent subsequence.

Note on *The subsequence obtained is not unique. We can, however, use the Bolzano-Weierstrass theorem to give a criterion for the convergence of a sequence in terms of the convergence of its subsequences.*

Theorem 3.10

If (x_n) is a bounded sequence of real numbers such that every convergent subsequence has the same limit x, then (x_n) converges to x.

4 Series

4.1 Convergence of series

Definition 4.1

Let (a_n) be a sequence of real numbers. The series $\sum_{n=1}^{\infty} a_n$ converges to a sum $S \subset \mathbb{R}$ if the sequence (S_n) of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

converges to S as $n \to \infty$. Otherwise, the series diverges.

Note on Example *If $|a| < 1$, the geometric series $\sum_{n=0}^{\infty} a^n$ converges; if $a \geq 1$, the geometric series diverges to ∞; if $a \leq -1$, the geometric series diverges in an oscillatory fashion.*

Proposition 4.1

A series $\sum a_n$ with positive terms $a_n \geq 0$ converges iff its partial sums

$$\sum_{k=1}^{n} a_k \leq M$$

are bounded from above, otherwise it diverges to ∞.

4.2 The Cauchy condition

Theorem 4.1 (Cauchy condition)

The series $\sum_{n=1}^{\infty} a_n$ covnerges iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=m+1}^{n} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon \quad \text{for all } n > m > N.$$

Theorem 4.2

If the series $\sum_{n=1}^{\infty} a_n$ covnerges, then

$$\lim_{n \to \infty} a_n = 0.$$

Note on *The condition is necessary, however, not sufficient to imply convergence. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even though $1/n \to 0$ as $n \to \infty$.*

4.3 Absolutely convergent series

Definition 4.2

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if

$$\sum_{n=1}^{\infty} |a_n| \text{ converges,}$$

and converges conditionally if

$$\sum_{n=1}^{\infty} a_n \text{ converges, but } \sum_{n=1}^{\infty} |a_n| \text{ diverges.}$$

Note on Example *The geometric series* $\sum a^n$ *is absolutely convergent if* $|a| < 1$, *and the alternating harmonic series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *is conditionally convergent.*

Definition 4.3

The positive and negative part of a real number $a \in \mathbb{R}$ *are given by*

$$a^+ = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a \leq 0, \end{cases} \qquad a^- = \begin{cases} 0 & \text{if } a \geq 0, \\ |a| & \text{if } a < 0. \end{cases}$$

Note on

$$a = a^+ - a^-, \quad |a| = a^+ + a^-.$$

Proposition 4.2

An absolutely convergent series converges. Moreover, $\sum_{n=1}^{\infty} a_n$ *converges absolutely iff the series* $\sum_{n=1}^{\infty} a_n^+$ *and* $\sum_{n=1}^{\infty} a_n^-$ *of positive and negative terms both converge. Furthermore, in that case*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-, \quad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-.$$

4.4 The comparison test

Theorem 4.3 (Comparison test)

Suppose that $b_n \geq 0$ *and* $\sum_{n=1}^{\infty} b_n$ *converges. If* $|a_n| \leq b_n$, *then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

4.5 The ratio and root tests

Theorem 4.4 (Ratio test)

Suppose that (a_n) *is a sequence of nonzero real numbers such that the limit*

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists or diverges to infinity. Then the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely if* $0 \leq r < 1$ *and diverges if* $1 < r \leq \infty$.

Theorem 4.5 (Root test)

Suppose that (a_n) *is a sequence of real numbers and let*

$$r = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely if* $0 \le r < 1$ *and diverges if* $1 < r \le \infty$.

Note on *The root test may succeed where the ratio test fails. For example, consider the geometric series with ratio* $1/2$, *i.e.,* $\sum_{n=1}^{\infty} \frac{1}{2^2}$. *Then both the ratio and root test imply convergence since*

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} < 1.$$

Now consider the series obtained by switching successive odd and even terms

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2^2} + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^6} + \dots, \qquad b_n = \begin{cases} 1/2^{n+1} & \text{if } n \text{ is odd} \\ 1/2^{n-1} & \text{if } n \text{ is even} \end{cases}$$

For this series,

$$\left|\frac{b_{n+1}}{b_n}\right| = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1/8 & \text{if } n \text{ is even,} \end{cases}$$

and the ratio test doesn't apply, since the required limit does not exist. On the other hand

$$\limsup_{n\to\infty} |b_n|^{1/n} = \frac{1}{2},$$

so the ratio test still works.

4.6 Alternating series

Definition 4.4

An alternating series is one in which successive terms have opposite signs.

Theorem 4.6 (Alternating series)

Suppose that (a_n) *is a decreasing sequence of nonnegative real numbers, meaning that* $0 \le a_{n+1} \le a_n$, *such that* $a_n \to 0$ *as* $n \to \infty$. *Then the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

converges.

4.7 Rearrangements

Definition 4.5 (Rearrangement)

A series $\sum_{m=1}^{\infty} b_m$ *is a rearrangement of a series* $\sum_{n=1}^{\infty} a_n$ *if there is a one-to-one, onto function* $f : \mathbb{N} \to \mathbb{N}$ *such that* $b_m = a_{f(m)}$.

Note on *If* $\sum b_m$ *is a rearrangement of* $\sum a_n$ *with* $n = f(m)$, *then* $\sum a_n$ *is a rearrangement of* $\sum b_m$, *with* $m = f^{-1}(n)$.

Theorem 4.7

If a series is absolutely convergent, then every rearrangement of the series converges to the same sum.

Theorem 4.8

If a series is conditionally convergent, then it has rearrangements that converge to an arbitrary real number and rearrangements that diverge to ∞ or $-\infty$.

4.8 The Cauchy product

Definition 4.6 (Cauchy product)

The Cauchy product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_k b_m = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right).$$

Theorem 4.9 (Cauchy product)

If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, then the Cauchy product is absolutely convergent and

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

5 Topology of the Real Numbers

5.1 Open sets

Definition 5.1 (Open set)

A set $G \subset \mathbb{R}$ is open if for every $x \in G$ there exists a $\delta > 0$ such that $G \supset (x - \delta, x + \delta)$.

Note on Example *The entire set of real numbers R is obviously open, and the empty set $\varnothing$ is open since it satisfies the definition vacuously (there is no $x \in \varnothing$). By contrast, the half-open interval $J = (0, 1]$ isn't open, since $1 \in J$ and $(1 - \delta, 1 + \delta)$ isn't a subset of J for any $\delta > 0$.*

Proposition 5.1

An arbitrary union of open sets is open, and a finite intersection of open sets is open.

Note on Example *The interval $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ is open for every $n \in \mathbb{N}$, but*

$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

is not open.

Definition 5.2

A set $U \subset \mathbb{R}$ is a neighborhood of a point $x \in \mathbb{R}$ if

$$U \supset (x - \delta, x + \delta)$$

for some $\delta > 0$. The open interval $(x - \delta, x + \delta)$ is called a δ-neighborbood of x.

Definition 5.3

A set $G \subset \mathbb{R}$ is open if every $x \in G$ has a neighborhood U such that $G \supset U$.

Proposition 5.2

A sequence (x_n) of real numbers converges to a limit $x \in \mathbb{R}$ iff for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

Definition 5.4 (Relatively open)

If $A \subset \mathbb{R}$ then $B \subset A$ is relatively open in A, or open in A, if $B = A \cap G$ where G is open in $\mathbb{R}$.

Note on Example *Let $A = [0, 1]$. Then the half-open intervals $(a, 1]$ and $[0, b)$ are open in A for every $0 \leq a < 1$ and $0 < b \leq 1$, since*

$$(a, 1] = [0, 1] \cap (a, 2), \quad [0, b) = [0, 1] \cap (-1, b)$$

and $(a, 2)$, $(-1, b)$ are open in $\mathbb{R}$.

Definition 5.5

If $A \subset \mathbb{R}$ then a relative neighborhood in A of a point $x \in A$ is a set $V = A \cap U$ where U is a neighborhood of x in $\mathbb{R}$.

Proposition 5.3

A set $B \subset A$ is relatively open in A iff every $x \in B$ has a relatively neighborhood V in A such that $B \supset V$.

5.2 Closed sets

Definition 5.6

A set $F \subset \mathbb{R}$ is closed if $F^c = \{x \in \mathbb{R} : x \notin F\}$ is open.

Note on *The empty set $\varnothing$ and $\mathbb{R}$ are both open and closed; they're the only such sets. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed. It isn't open because every neighborhood of a rational number contains irrational numbers, and its complement isn't open because every neighborhood of an irrational number contains rational numbers.*

Proposition 5.4

A set $F \subset \mathbb{R}$ is closed iff the limit of every convergent sequence in F belongs to F.

Note on Example *Suppose that (x_n) is a convergent sequence in $[0,1]$. Then $0 \leq x_n \leq 1$ for all $n \in \mathbb{N}$, and since limits preserve (non-strict) inequalities, we have*

$$0 \leq \lim_{n\to\infty} x_n \leq 1,$$

meaning that the limit belongs to $[0,1]$. On the other hand, the half-open interval $I = (0,1]$ isn't closed since, for example, $(1/n)$ is a convergent sequence in I whose limit 0 doesn't belong to I.

Proposition 5.5

An arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed.

Note on Example *If I_n is the closed interval $I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, then the union of the I_n is an open interval*

$$\bigcup_{n=1}^{\infty} I_n = (0,1).$$

Definition 5.7

Let $A \subset \mathbb{R}$ be a subset of $\mathbb{R}$. Then $x \in \mathbb{R}$ is:

(1) an interior point of A if there exists $\delta > 0$ such that $A \supset (x-\delta, x+\delta)$;

(2) an isolated point of A if $x \in A$ and there exists $\delta > 0$ such that x is the only point in A that belongs to the interval $(x-\delta, x+\delta)$;

(3) a boundary point of A if for every $\delta > 0$ the interval $(x-\delta, x+\delta)$ contains points in A and points not in A;

(4) an accumulation point of A if for every $\delta > 0$ the interval $(x-\delta, x+\delta)$ contains a point in A that is distinct from x.

Note on Example *Let $I = (a,b)$ be an open interval and $J = [a,b]$ a closed interval. Then the set of interior points of I or J is (a,b), and the set of boundary points consists of the two endpoints $\{a,b\}$. The set of accumulation points of I or J is the closed interval $[a,b]$ and I, J have no isolated points.*

Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then every point of A is an isolated point.

Every point of $\mathbb{N}$ is both a boundary point and an isolated point. The set $\mathbb{Q}$ has no interior or isolated points, and every real number is both a boundary and accumulation point of $\mathbb{Q}$.

Proposition 5.6

A point $x \in \mathbb{R}$ is an accumulation point of $A \subset \mathbb{R}$ iff there is a sequence (x_n) in A with $x_n \neq x$ for every $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$.

Note on Example *If $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, then 0 is an accumulation point of A, since $(1/n)$ is a sequence in A such that $1/n \to 0$ as $n \to \infty$. On the other hand, 1 is not an accumulation point of A since the only sequence in A that converges to 1 are the ones whose terms eventually equal 1, and the therms are required to be distinct from 1.*

Proposition 5.7

A set $A \subset \mathbb{R}$ is:

(1) iff every point of A is an interior point;

(2) iff every accumulation point belongs to A.

5.3 Compact sets

Definition 5.8 (Sequentially compact)

A set $K \subset \mathbb{R}$ is sequentially compact if every sequence in K has a convergent subsequence whose limit belongs to K.

Note on Example *The open interval $I = (0, 1)$ is not compact. The sequence $(1/n)$ in I converges to 0. The set $\mathbb{N}$ is closed, but it is not compact. The sequence (n) in $\mathbb{N}$ has no convergent subsequence.*

Theorem 5.1 (Bolzano-Weierstrass)

A subset of $\mathbb{R}$ is sequentially compact iff it is closed and bounded.

Note on Example *Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then A is not compact, since it isn't closed. However, the set $K = A \cup \{0\}$ is closed and bounded, so it is compact.*

Proposition 5.8

If $K \subset \mathbb{R}$ is compact, then K has a maximum and minimum.

Theorem 5.2

Let $\{K_n : n \in \mathbb{N}\}$ be a decreasing sequence of nonempty compact sets of real numbers, meaning that

$$K_1 \supset K_2 \supset \cdots \supset K_n \supset K_{n+1} \supset \ldots,$$

and $K_n \neq \varnothing$. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

Moreover, if $\operatorname{diam} K_n \to 0$ as $n \to \infty$, then the intersection consists of a single point.

Note on *We refer to a decreasing sequence of sets as a nested sequence. In the case when each $K_n = [a_n, b_n]$ is a compact interval, the preceding result is called the nested interval theorem. For example, the nested compact intervals $[0, 1 + 1/n]$ have nonempty intersection $[0, 1]$. Here diam$[0, 1 + 1/n] \to 1$ as $n \to \infty$, and the intersection consists of an interval. The nested compact intervals $[0, 1/n]$ have nonempty intersection $\{0\}$, since diam$[0, 1/n] \to 0$ as $n \to \infty$.*

Define a nested sequence $A_1 \supset A_2 \supset \ldots$ of non-compact sets by

$$A_n = \left\{\frac{1}{k} : k = n, n+1, n+2, \ldots\right\},$$

Then $\bigcap_{n=1}^{\infty} A_n = \varnothing$. *However, if we add* 0 *to the* A_n *to make them compact and define* $K_n = A_n \cup \{0\}$, *then the intersection*

$$\bigcap_{n=1}^{\infty} K_n = \{0\}$$

is nonempty.

Definition 5.9 ((Open) Cover)

Let $A \subset \mathbb{R}$.

(1) *A cover of* A *is a collection of sets* $\{A_i \subset \mathbb{R} : i \in I\}$ *whose union contains* A,

$$\bigcup_{i \in I} A_i \supset A.$$

(2) *An open cover of* A *is a cover such that* A_i *is open for every* $i \in I$.

Note on Example *Let* $A_i = (1/i, 2)$. *Then* $\mathcal{C} = \{A_i : i \in \mathbb{N}\}$ *is an open cover of* $(0, 1]$, *since*

$$\bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 2\right) = (0, 2) \supset (0, 1].$$

On the other hand, $\mathcal{C}$ *is not a cover of* $[0, 1]$ *since its union does not contain* 0.

If $A_i = (i - 1, i + 1)$, *then* $\{A_i : i \in \mathbb{Z}\}$ *is an open cover of* $\mathbb{R}$. *On the other hand, if* $B_i = (i, i + 1)$, *then* $\{B_i : i \in \mathbb{Z}\}$ *is not open cover of* $\mathbb{R}$, *since its union doesn't contain any of the integers. Finally, if* $C_i = [i, i + 1)$, *then* $\{C_i : i \in \mathbb{Z}\}$ *is a cover of* $\mathbb{R}$ *by disjoint, half-open intervals, but it isn't an open cover.*

Definition 5.10

Suppose that $\mathcal{C} = \{A_i \subset \mathbb{R} : i \in I\}$ *is a cover of* $A \subset \mathbb{R}$.

(1) *A subcover* $\mathcal{S}$ *of* $\mathcal{C}$ *is a sub-collection* $\mathcal{S} \subset \mathcal{C}$ *that covers* A, *meaning that*

$$\mathcal{S} = \{A_{i_k} \in \mathcal{C} : k \in J\}, \quad \bigcup_{k \in J} A_{i_k} \supset A.$$

(2) *A finite subcover is a subcover* $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ *that consists of finitely many sets.*

Note on Example *Let* $A_i = (1/i, 2)$. *Then* $\mathcal{C} = \{A_i : i \in \mathbb{N}\}$ *is an open cover of* $(0, 1]$, *since*

$$\bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 2\right) = (0, 2) \supset (0, 1].$$

Then $\{A_{2j} : j \in \mathbb{N}\}$ *is a subcover. However, there is no finite subcover. On the other hand, the cover* $\mathcal{C}' = \mathcal{C} \cup \{(-\delta, \delta)\}$ *of* $[0, 1]$ *does have a finite subcover.*

Definition 5.11

A set $K \subset \mathbb{R}$ *is compact if every open cover of* K *has a finite subcover.*

Note on Example *The collection of open intervals*

$$\{A_i : i \in \mathbb{N}\}, \quad A_i = (i - 1, i + 1)$$

is an open cover of the natural numbers $\mathbb{N}$, since

$$\bigcup_{i=1}^{\infty} A_i = (0, \infty) \supset \mathbb{N}.$$

However, no finite sub-collection $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ covers $\mathbb{N}$. Thus, $\mathbb{N}$ is not compact.

Theorem 5.3 (Heine-Borel)

A subset of $\mathbb{R}$ is compact iff it is closed and bounded.

Corollary 5.1

A subset of $\mathbb{R}$ is compact iff it is sequentially compact.

Note on *This can be generalizes to an arbitrary metric space.*

5.4 Connected sets

Definition 5.12

(1) A set of real numbers $A \subset \mathbb{R}$ is disconnected if there are disjoint open sets $U, V \subset \mathbb{R}$ such that $A \cap U$ and $A \cap V$ are nonempty and

$$A = (A \cap U) \cup (A \cap V).$$

(2) A set is connected if it not disconnected.

Note on Example *The set $\{0, 1\}$ consisting of two points is disconnected. For example, $U = (-1/2, 1/2)$ and $V = (1/2, 3/2)$ satisfy the definition.*

The set $\mathbb{Q}$ of rational numbers is disconnected.

Definition 5.13

A set of real numbers $I \subset \mathbb{R}$ is an interavl if $x, y \in I$ and $x < y$ implies that $z \in I$ for every $x < z < y$.

Note on Example *For example, (a, b), $(a, b]$ or $[a, b]$, if $a = b$, then $[a, a] = \{a\}$ is an interval that consists of a single point.*

Theorem 5.4

A set of real numbers is connected iff it is an interval.

5.5 The Cantor set

Definition 5.14

We define a nested sequence (F_n) of sets $F_n \subset [0, 1]$ as follows. First, we remove the middle-third from $[0, 1]$ to get F_1:

$$F_1 = I_0 \cup I_1, \quad I_0 = \left[0, \frac{1}{3}\right], \quad I_1 = \left[\frac{2}{3}, 1\right].$$

Next, we remove middle-thirds from I_0 and I_1 and derive I_{00}, I_{01}, I_{10} and I_{11}. Continuing *in this way, we get at the nth stage a set of the form*

$$F_n = \bigcup_{\mathbf{s}\in\Sigma_n} I_{\mathbf{s}},$$

where $\Sigma_n = \{(s_1, s_2, \ldots, s_n) : s_n = 0, 1\}$ *is the set of binary n-tuples. Furthermore, each* $I_{\mathbf{s}} = [a_{\mathbf{s}}, b_{\mathbf{s}}]$ *is a closed interval, and if* $\mathbf{s} = (s_1, s_2, \ldots, s_n)$*, then*

$$a_{\mathbf{s}} = \sum_{k=1}^{n} \frac{2s_k}{3^k}, \quad b_{\mathbf{s}} = a_{\mathbf{s}} + \frac{1}{3^n}.$$

The Cantor set C is the intersection

$$C = \bigcap_{n=1}^{\infty} F_n,$$

of the nested sequence of sets (F_n).

Note on *The Cantor set C is clearly nonempty since the endpoints a_s, b_s are contained in F_n. These endpoints form a countably infinite set. What may be initially suprising is that there are uncountably many other points in C that are not endpoints. For example, $1/4$ is not one of the endpoints, but it belongs to C because it has a base three expansion consisting entirely of 0's and 2's.*

Theorem 5.5

The Cantor set is compact.

Proof The Cantor set C is bounded, since it is a subset of $[0, 1]$. All the sets F_n are closed, their intersection is closed. ■

Theorem 5.6

The Cantor set has the same cardinality as Σ.

Theorem 5.7

The set $\mathbb{R}$ of real numbers has the same cardinality as $\mathcal{P}(\mathbb{N})$.

6 Limits of Functions

6.1 Limits

Definition 6.1

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Proposition 6.1

The limit of a function is unique if it exists.

Definition 6.2

A set $U \subset \mathbb{R}$ is a punctured (or deleted) neighborhood of $c \in \mathbb{R}$ if $U \supset (c-\delta, c) \cup (c, c+\delta)$ for some $\delta > 0$. The set $(c-\delta, c) \cup (c, c+\delta)$ is called a punctured (or delted) δ-neighborhood of c.

Definition 6.3

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then $\lim_{x \to c} f(x) = L$ iff for every neighborhood V of L, there is a punctured neighborhood U of c such that

$$x \in A \cap U \text{ implies that } f(x) \in V.$$

Note on *This is essentially a rewording of the $\epsilon - \delta$ definition.*

Theorem 6.1

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then $\lim_{x \to c} f(x) = L$ iff

$$\lim_{n \to \infty} f(x_n) = L$$

for every sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} x_n = c.$$

Corollary 6.1

Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of A. Then $\lim_{x \to c} f(x)$ does not exist if either of the following conditions holds:

(1) There are sequences (x_n), (y_n) in A with $x_n, y_n \neq c$ such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c, \quad \text{but} \quad \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n).$$

(2) There is a sequence (x_n) in A with $x_n \neq c$ such that $\lim_{n \to \infty} x_n = c$ but the sequence $(f(x_n))$ diverges.

Note on Example

(1) Define the sign function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ by following. Then, the limit $\lim_{x \to 0} \operatorname{sgn} x$ doesn't exist.

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(2) The limit $\lim_{x \to 0} \frac{1}{x}$ corresponding to the function $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$, doesn't exist.

Definition 6.4

If $f : A \to \mathbb{R}$ is a real-valued function, then

$$\sup_A f = \sup\{f(x) : x \in A\}, \quad \inf_A f = \inf\{f(x) : x \in A\}.$$

Definition 6.5

If $f : A \to \mathbb{R}$, then f is bounded from above if $\sup_A f$ is finite, bounded from below if $\inf_A f$ is finite, and bounded if both are finite. A function that is not bounded is said to be unbounded.

Definition 6.6

Suppose that $f : A \to \mathbb{R}$ and c is an accumulation point of A. Then f is locally bounded at c if there is a neighborhood U of c such that f is bounded on $A \cap U$.

Note on Example *The function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is locally bounded at every $0 < c \leq 1$, but it is not locally bounded at 0.*

Proposition 6.2

Suppose that $f : A \to \mathbb{R}$ and c is an accumulation point of A. If $\lim_{x\to c} f(x)$ exists, then f is locally bounded at c.

Note on Example *The function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by*

$$f(x) = \sin\left(\frac{1}{x}\right)$$

is bounded, but $\lim_{x\to 0} f(x)$ doesn't exist.

6.2 Left, right, and infinite limits

Definition 6.7 (Right and left limits)

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$.

(1) If $c \in \mathbb{R}$ is an accumulation point of $\{x \in A : x > c\}$, then f has the right limit $\lim_{x\to c^+} f(x) = L$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$c < x < c + \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

(2) If $c \in \mathbb{R}$ is an accumulation point of $\{x \in A : x < c\}$, then f has the left limit $\lim_{x\to c^-} f(x) = L$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$c - \delta < x < c \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Proposition 6.3

Suppose that $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and $c \in \mathbb{R}$ is an accumulation point of both $\{x \in A : x > c\}$ and $\{x \in A : x < c\}$. Then $\lim_{x\to c} f(x) = L$ iff

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L.$$

Definition 6.8 (Limits as $x \to \pm\infty$)

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$.

(1) *If A is not bounded from above, then $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ there exists an $M \in \mathbb{R}$ such that*

$$x > M \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

(2) *If A is not bounded from below, then $\lim_{x\to-\infty} f(x) = L$ if for every $\epsilon > 0$ there exists an $m \in \mathbb{R}$ such that*

$$x < m \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Definition 6.9 (Divergence to $\pm\infty$)

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

(1) *$\lim_{x\to c} f(x) = \infty$ if for every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that*

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } f(x) > M,$$

(2) *and $\lim_{x\to c} f(x) = -\infty$ if for every $m \in \mathbb{R}$ there exists a $\delta > 0$ such that*

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } f(x) < m.$$

6.3 Properties of limits

Theorem 6.2 (Order properties)

Suppose that $f, g : A \to \mathbb{R}$ and c is an accumulation point of A. If

$$f(x) \le g(x) \quad \text{for all } x \in A,$$

and $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

Theorem 6.3 ("Sandwich" or "Squeeze" criterion)

Suppose that $f, g, h : A \to \mathbb{R}$ and c is an accumulation point of A. If

$$f(x) \le g(x) \le h(x) \quad \text{for all } x \in A,$$

and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L,$$

then the limit of $g(x)$ as $x \to c$ exists and

$$\lim_{x\to c} g(x) = L.$$

Theorem 6.4 (Algebraic properties)

Suppose that $f, g : A \to \mathbb{R}$, c is an accumulation point of A, and the limits

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M$$

exist. Then

$$\lim_{x \to c} kf(x) = kL \quad \text{for every } k \in \mathbb{R}$$
$$\lim_{x \to c} [f(x) + g(x)] = L + M$$
$$\lim_{x \to c} [f(x)g(x)] = LM,$$
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{if } M \neq 0$$

7 Continuous Functions

7.1 Continuity

Definition 7.1

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in A$. Then f is continuous at c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - f(c)| < \epsilon.$$

A function $f : A \to \mathbb{R}$ is continuous if it is continuous at every point of A.

Definition 7.2

A function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$ is continuous at $c \in A$ if for every neighborhood V of $f(c)$ there is a neighborhood U of c such that

$$x \in A \cap U \text{ implies that } f(x) \in V.$$

Note on *The $\epsilon - \delta$ definition corresponds to the case when V is an ϵ-neighborhood of $f(c)$ and U is a δ-neighborhood of c.*

Note on *Note that c must belong to the domain A of f in order to define the continuity of f at c. If c is an isolated point of A, then the continuity condition holds automatically. If $c \in A$ is an accumulation point of A, then the continuity of f at c is equivalent to the condition that*

$$\lim_{x \to c} f(x) = f(c).$$

For example, suppose that $A = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ and $f : A \to \mathbb{R}$ is defined by

$$f(0) = y_0, \quad f\left(\frac{1}{n}\right) = y_n.$$

Then $1/n$ is an isolated point of A, so f is continuous at $1/n$ for every choice of y_n. The remaining point $0 \in A$ is an accumulation point of A, and the condition for f to be continuous at 0 is that

$$\lim_{n \to \infty} y_n = y_0.$$

Theorem 7.1

If $f : A \to \mathbb{R}$ and $c \in A$ is an accumulation point of A, then f is continuous at c iff

$$\lim_{n\to\infty} f(x_n) = f(c)$$

for every sequence (x_n) in A such that $x_n \to c$ as $n \to \infty$.

Note on Example *The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous at every $c \in \mathbb{R}$.

The Thomae function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbb{Q} \text{ where } p \text{ and } q > 0 \text{ are relatively prime,} \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0. \end{cases}$$

is continuous at 0 and every irrational number and discontinuous at every nonzero rational number.

Definition 7.3 (Discontinuity)

We give a rough classification of discontinuities of a funciton $f : A \to \mathbb{R}$ at an accumulation point $c \in a$ as follows.

(1) ***Removable discontinuity.*** $\lim_{x\to c} f(x) = L$ *exists but* $L \neq f(c)$.

(2) ***Jump discontinuity.*** $\lim_{x\to c} f(x)$ *doesn't exist, but both the left and right limits exist and are different.*

(3) ***Essential discontinuity.*** $\lim_{x\to c} f(x)$ *doesn't exist and at least one of the left or right limits doesn't exist.*

7.2 Properties of continuous functions

Theorem 7.2

If $f, g : A \to \mathbb{R}$ are continuous at $c \in A$ and $k \in \mathbb{R}$, then kf, $f + g$, and fg are continuous at c. Moreover, if $g(c) \neq 0$ then f/g is continuous at c.

Corollary 7.1

Every polynomial function is continuous on $\mathbb{R}$ and every rational function is continuous on its domain.

Theorem 7.3

Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ where $f(A) \subset B$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f : A \to \mathbb{R}$ is continuous at c.

Corollary 7.2

Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ where $f(A) \subset B$. If f is continuous on $f(A)$, then $g \circ f$ is continuous on A.

7.3 Uniform continuity

Definition 7.4

Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$. Then f is uniformly continuous on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| < \delta \text{ and } x, y \in A \text{ implies that } |f(x) - f(y)| < \epsilon.$$

Note on *The key point of this definition is that δ depends only on ϵ, not on x, y. A uniformly continuous function on A is continuous at every point of A, but the converse is not true.*

Proposition 7.1

A function $f : A \to \mathbb{R}$ is not uniformly continuous on A iff there exists $\epsilon > 0$ and sequences (x_n), (y_n) in A such that

$$\lim_{n\to\infty} |x_n - y_n| = 0 \text{ and } |f(x_n) - f(y_n)| \geq \epsilon_0 \text{ for all } n \in \mathbb{N}.$$

Note on *The sine function is uniformly continuous on $\mathbb{R}$, since we can take $\delta = \epsilon$ for every $x, y \in \mathbb{R}$.*

Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$. Then f is uniformly continuous on $[0, 1]$ since we can take $\delta = \epsilon/2$. However, f is continuous but not uniformly continuous on $\mathbb{R}$. Let

$$x_n = n, \quad y_n = n + \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

but

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 \quad \text{for every } n \in \mathbb{N}.$$

The function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous on $(0, 1]$. We define $x_n, y_n \in (0, 1]$ for $n \in \mathbb{N}$ by

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n+1}.$$

Then $|x_n - y_n| \to 0$ as $n \to \infty$, but

$$|f(x_n) - f(y_n)| = (n + 1) - n = 1 \quad \text{for every } n \in \mathbb{N}.$$

The non-uniformly continuous functions in the last two examples were unbounded. However, even bounded continuous functions can fail to be uniformly continuous if they oscillate arbitrarily quickly. For example, define $f : (0, 1] \to \mathbb{R}$ by $f(x) = \sin\left(\frac{1}{x}\right)$. Then f is continuous on $(0, 1]$

but it isn't uniformly continuous on $(0, 1]$. *Define* $x_n, y_n \in (0, 1]$ *for* $n \in \mathbb{N}$ *by*

$$x_n = \frac{1}{2n\pi}, \quad y_n = \frac{1}{2n\pi + \pi/2}.$$

Then $|x_n - y_n| \to 0$ *as* $n \to \infty$, *but*

$$|f(x_n) - f(y_n)| = \sin\left(2n\pi + \frac{\pi}{2}\right) - \sin 2n\pi = 1 \quad \text{for all } n \in \mathbb{N}.$$

7.4 Continuous functions and open sets

Theorem 7.4

A function $f : A \to \mathbb{R}$ *is continuous on* A *iff* $f^{-1}(V)$ *is open in* A *for every set* V *that is open in* $\mathbb{R}$.

Note on *A continuous function needn't map open sets to open sets. However, the inverse image of an open set under a continuous function is always open. For example, define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x^2$, *consider the open interval* $I = (1, 4)$. *Then both* $f(I) = (1, 16)$ *and* $f^{-1}(I) = (-2, -1) \cup (1, 2)$ *are open. On the other hand, if* $J = (-1, 1)$, *then*

$$f(J) = [0, 1), \quad f^{-1}(J) = (-1, 1),$$

so the inverse image of the open interval J *is open, but the image is not.*

Theorem 7.5

Suppose that $f : I \to \mathbb{R}$ *is continuous and* $I \subset \mathbb{R}$ *is an interval. Then* $f(I)$ *is an interval.*

Definition 7.5

An open mapping on a set $A \subset \mathbb{R}$ *is a function* $f : A \to \mathbb{R}$ *such that* $f(B)$ *is open in* $\mathbb{R}$ *for every set* $B \subset A$ *that is open in* A.

Note on *A continuous function needn't be open, but if* $f : A \to \mathbb{R}$ *is continuous and one-to-one, then* $f^{-1} : f(A) \to \mathbb{R}$ *is open. For example,* $f(x) = x^2$ *is not an open mapping on* $\mathbb{R}$, *while* $f : [0, \infty) \to \mathbb{R}$ *is open because it is one-to-one with a continuous inverse* $f^{-1} : [0, \infty) \to \mathbb{R}$ *given by* $f^{-1}(x) = \sqrt{x}$.

7.5 Continuous functions on compact sets

Theorem 7.6

If $K \subset \mathbb{R}$ *is compact and* $f : K \to \mathbb{R}$ *is continuous, then* $f(K)$ *is compact.*

Note on *Note that compactness is essential here; it is not true, in general, that a continuous function maps closed sets to closed sets. For example, define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = \frac{1}{1+x^2}$. *Then* $[0, \infty)$ *is closed but* $f([0, \infty)) = (0, 1]$ *is not.*

Theorem 7.7 (Weierstrass extreme value)

If $f : K \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is compact, then f is bounded on K and f attains its maximum and minimum values on K.

Theorem 7.8

If $f : K \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is compact, then f is uniformly continuous on K.

7.6 The intermediate value theorem

Theorem 7.9 (Intermediate value)

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed, bounded interval. If $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$, then there is a point $a < c < b$ such that $f(c) = 0$.

Theorem 7.10 (Intermediate value theorem)

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed, bounded interval. Then for every d strictly between $f(a)$ and $f(b)$ there is a point $a < c < b$ such that $f(c) = d$.

Theorem 7.11

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed, bounded interval. Then $f([a, b]) = [m, M]$ is a closed, bounded interval.

7.7 Monotonic functions

Definition 7.6

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is

(1) increasing if

$$f(x_1) \leq f(x_2) \qquad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2,$$

(2) strictly increasing if

$$f(x_1) < f(x_2) \qquad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2.$$

An increasing or decreasing function is called a monotonic function, and a strictly increasing or strictly decreasing function is called a strictly monotonic function.

Theorem 7.12

If $f : I \to \mathbb{R}$ is monotonic on an interval I, then the left and right limits of f,

$$\lim_{x \to c^-} f(x), \qquad \lim_{x \to c^+} f(x),$$

exist at every interior point c of I.

Corollary 7.3

Every discontinuity of a monotonic function $f : I \to \mathbb{R}$ at an interior point of the interval I is a jump discontinuity.

8 Differentiable Functions

8.1 The derivative

Definition 8.1

Suppose that $f : (a, b) \to \mathbb{R}$ and $a < c < b$. Then f is differentiable at c with derivative $f'(c)$ if

$$\lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\right] = f'(c).$$

Proposition 8.1

Suppose that $f : (a, b) \to \mathbb{R}$. Then f is differentiable at $c \in (a, b)$ iff there exists a constant $A \in \mathbb{R}$ and a function $r : (a - c, b - c) \to \mathbb{R}$ such that

$$f(c+h) = f(c) + Ah + r(h), \qquad \lim_{h\to 0}\frac{r(h)}{h} = 0.$$

In that case, $A = f'(c)$.

Definition 8.2

Suppose that $f : [a, b] \to \mathbb{R}$. Then f is right-differentiable at $a \le c < b$ with right derivative $f'(c^+)$ if

$$\lim_{h\to 0^+}\left[\frac{f(c+h)-f(c)}{h}\right] = f'\left(c^+\right)$$

exists, and f is left-differentiable at $a < c \le b$ with left derivative $f'(c^-)$ if

$$\lim_{h\to 0^-}\left[\frac{f(c+h)-f(c)}{h}\right] = \lim_{h\to 0^+}\left[\frac{f(c)-f(c-h)}{h}\right] = f'\left(c^-\right).$$

8.2 Properties of the derivative

Theorem 8.1

If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then f is continuous at c.

Definition 8.3 (Continuous differentiability)

A function $f : (a, b) \to \mathbb{R}$ is continuously differentiable on (a, b), written $f \in C^1(a, b)$, if it is differentiable on (a, b) and $f' : (a, b) \to \mathbb{R}$ is continuous.

Theorem 8.2 (Algebraic properties)

If $f, g : (a, b) \to \mathbb{R}$ are differentiable at $c \in (a, b)$ and $k \in \mathbb{R}$, then kf, $f + g$, and fg are differentiable at c with

$$(kf)'(c) = kf'(c), \quad (f+g)'(c) = f'(c) + g'(c), \quad (fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

Furthermore, if $g(c) \neq 0$, then f/g is differentiable at c with

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}.$$

8.3 The chain rule

Theorem 8.3 (Chain rule)

Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ where $A \subset \mathbb{R}$ and $f(A) \subset B$, and suppose that c is an interior point of A and $f(c)$ is an interior point of B. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f : A \to \mathbb{R}$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proposition 8.2

Suppose that $f : A \to \mathbb{R}$ is a one-to-one function on $A \subset \mathbb{R}$ with inverse $f^{-1} : B \to \mathbb{R}$ where $B = f(A)$. Assume that f is differentiable at an interior point $c \in A$ and f^{-1} is differentiable at $f(c)$, where $f(c)$ is an interior point of B. Then $f'(c) \neq 0$ and

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}.$$

8.4 Extreme values

Definition 8.4

Suppose that $f : A \to \mathbb{R}$.

(1) Then f has a global (or absolute) maximum at $c \in A$ if

$$f(x) \le f(c) \quad \text{for all } x \in A,$$

(2) and f has a local (or relative) maximum at $c \in A$ if there is a neighborhood U of c such that

$$f(x) \le f(c) \quad \text{for all } x \in A \cap U.$$

Theorem 8.4

If $f : A \subset \mathbb{R} \to \mathbb{R}$ has a local extreme value at an interior point $c \in A$ and f is differentiable at c, then $f'(c) = 0$.

Proposition 8.3

Let $f : [a, b] \to \mathbb{R}$. If the right derivative of f exists at a, then:

(1) $f'(a^+) \leq 0$ if f has a local maximum at a;

(2) and $f'(a^+) \geq 0$ if f has a local minimum at a.

Definition 8.5

Suppose that $f : A \subset \mathbb{R} \to \mathbb{R}$.

(1) An interior point $c \in A$ such that f is not differentiable at c or $f'(c) = 0$ is called a critical point of f.

(2) An interior point where $f'(c) = 0$ is called a stationary point of f.

8.5 The mean value theorem

Theorem 8.5 (Rolle)

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on the closed, bounded interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there exists $a < c < b$ such that $f'(c) = 0$.

Theorem 8.6 (Mean value)

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on the closed, bounded interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 8.7

If $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and $f'(x) = 0$ for every $a < x < b$, then f is constant on (a, b).

Corollary 8.1

If $f, g : (a, b) \to \mathbb{R}$ are differentiable on (a, b) and $f'(x) = g'(x)$ for every $a < x < b$, then $f(x) = g(x) + C$ for some constant C.

Theorem 8.8

Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then f is increasing iff $f'(x) \geq 0$ for every $a < x < b$, and decreasing iff $f'(x) \leq 0$ for every $a < x < b$.

Note on *Note that although $f' > 0$ implies that f is strictly increasing, f is strictly increasing does not imply that $f' > 0$. For example, $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.*

8.6 Taylor's theorem

Definition 8.6

Let $f : (a, b) \to \mathbb{R}$ and suppose that f has n derivatives

$$f', f'', \dots f^{(n)} : (a, b) \to \mathbb{R}$$

on (a, b). The Taylor polynomial of degree n of f at $a < c < b$ is

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{1}{2!} f''(c)(x - c)^2 + \cdots + \frac{1}{n!} f^{(n)}(c)(x - c)^n.$$

Equivalently,

$$P_n(x) = \sum_{k=0}^{n} a_k (x - c)^k, \quad a_k = \frac{1}{k!} f^{(k)}(c).$$

Theorem 8.9 (Taylor with Lagrange Remainder)

Suppose that $f : (a, b) \to \mathbb{R}$ has $n + 1$ derivatives on (a, b) and let $a < c < b$. For every $a < x < b$, there exists ξ between c and x such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{1}{2!} f''(c)(x - c)^2 + \cdots + \frac{1}{n!} f^{(n)}(c)(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{(n + 1)!} f^{(n+1)}(\xi)(x - c)^{n+1}.$$

8.7 The inverse function theorem

Definition 8.7

A function $f : A \to \mathbb{R}$ is locally invertible at an interior point $c \in A$ if there exists open neighborhood U of c and V of $f(c)$ such that $f|_U : U \to V$ is one-to-one and onto, in which case f has a local inverse $(f|_U)^{-1} : V \to U$.

Theorem 8.10 (Inverse function)

Suppose that $f : A \subset \mathbb{R} \to \mathbb{R}$ and $c \in A$ is an interior point of A. If f is differentiable in a neighborhood of c, $f'(c) \neq 0$, and f' is continuous at c, then there are open neighborhoods U of c and V of $f(c)$ such that f has a local inverse $(f|_U)^{-1} : V \to U$. Furthermore, the local inverse function is differentiable at $f(c)$ with derivative

$$\left[(f|_U)^{-1}\right]'(f(c)) = \frac{1}{f'(c)}.$$

8.8 L'Hôspital's rule

Theorem 8.11 (Cauchy mean value)

Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous on the closed, bounded interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists $a < c < b$ such that

$$f'(c)[g(b) - g(a)] = [f(b) - f(a)]g'(c).$$

Theorem 8.12 (L'Hôspital's rule: $0/0$)

Suppose that $f, g : [a, b] \to \mathbb{R}$ are differentiable functions on a bounded open interval (a, b) such that $g'(x) \neq 0$ for $x \in (a, b)$ and

$$\lim_{x \to a^+} f(x) = 0, \qquad \lim_{x \to a^+} g(x) = 0.$$

Then

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \quad \textit{implies that} \quad \lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

Theorem 8.13 (L'Hôspital's rule: ∞/∞)

Suppose that $f, g : [a, b] \to \mathbb{R}$ are differentiable functions on a bounded open interval (a, b) such that $g'(x) \neq 0$ for $x \in (a, b)$ and

$$\lim_{x \to a^+} |g(x)| = \infty.$$

Then

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \quad \textit{implies that} \quad \lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

9 Sequences and Series of Functions

9.1 Pointwise convergence

Definition 9.1

Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ pointwise on A if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in A$.

We say that the sequence (f_n) converges pointwise if it converges pointwise to some function f, in which case

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Note on Example *Pointwise convergence does not, in general, preserve boundedness. For example, suppose that $f_n : (0, 1) \to \mathbb{R}$ is defined by $f_n(x) = \frac{n}{nx+1}$. Then, since $x \neq 0$,*

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x + 1/n} = \frac{1}{x} = f(x).$$

We have $|f_n(x)| < n$ for all $x \in (0, 1)$, so each f_n is bounded on $(0, 1)$, but the pointwise limit f is not.

Pointwise convergence does not, in general, preserve continuity. For example, suppose that $f_n : [0,1] \to \mathbb{R}$ is defined by $f_n(x) = x^n$. So $f_n \to f$ pointwise where

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Although each f_n is continuous on $[0,1]$, the pointwise limit f is not.

A pointwise convergent sequence (f_n) of functions need not be uniformly bounded. For example, suppose that $f_n : [0,1] \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 2n^2 x & \text{if } 0 \le x \le 1/(2n) \\ 2n^2(1/n - x) & \text{if } 1/(2n) < x < 1/n \\ 0 & 1/n \le x \le 1 \end{cases}$$

So $f_n \to 0$ pointwise on $[0,1]$.

In general, one cannot differentiate a pointwise convergent sequence. Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{\sin nx}{n}$. Then $f_n \to 0$ pointwise on $\mathbb{R}$. The sequence (f_n') of derivatives $f_n'(x) = \cos nx$ does not converge pointwise on $\mathbb{R}$; for example, $f_n'(\pi) = (-1)^n$ does not converge as $n \to \infty$.

9.2 Uniform convergence

Definition 9.2

Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ uniformly on A if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \text{ implies that } |f_n(x) - f(x)| < \epsilon \text{ for all } x \in A.$$

Note on *The crucial point in this definition is that N depends only on ϵ and not on $x \in A$, whereas for a pointwise convergent sequence N may depend on both ϵ and x. A uniformly convergent sequence is always pointwise convergent (to the same limit), but the converse is not true.*

Note on Example *The sequence $f_n(x) = x^n$ converges pointwise on $[0,1]$ but not uniformly on $[0,1]$. For $0 \le x < 1$, we have*

$$|f_n(x) - f(x)| = x^n.$$

If $0 < \epsilon < 1$, we cannot make $x^n < \epsilon$ for all $0 \le x < 1$ however large we choose n. The problem is that x^n converges to 0 at an arbitrarily slow rate for x sufficiently close to 1.

The pointwise convergent sequence $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 2n^2 x & \text{if } 0 \le x \le 1/(2n) \\ 2n^2(1/n - x) & \text{if } 1/(2n) < x < 1/n \\ 0 & 1/n \le x \le 1 \end{cases}$$

does not converge uniformly.

The functions $f_n : \mathbb{R} \to \mathbb{R}$ *by* $f_n(x) = \frac{\sin nx}{n}$ *converge uniformly to* 0 *on* $\mathbb{R}$.

9.3 Cauchy condition for uniform convergence

Definition 9.3

A sequence (f_n) *of functions* $f_n : A \to \mathbb{R}$ *is uniformly Cauchy on* A *if for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$m, n > N \text{ implies that } |f_m(x) - f_n(x)| < \epsilon \text{ for all } x \in A.$$

Theorem 9.1

A sequence (f_n) *of functions* $f_n : A \to \mathbb{R}$ *converges uniformly on* A *iff it is uniformly Cauchy on* A.

9.4 Properties of uniform convergence

Theorem 9.2 (Boundedness)

Suppose that $f_n : A \to \mathbb{R}$ *is bounded on* A *for every* $n \in \mathbb{N}$ *and* $f_n \to f$ *uniformly on* A. *Then* $f : A \to \mathbb{R}$ *is bounded on* A.

Note on *In particular, it follows that if a sequence of bounded functions converges pointwise to an unbounded function, then the convergence is not uniform.*

Theorem 9.3 (Continuity)

If a sequence (f_n) *of continuous functions* $f_n : A \to \mathbb{R}$ *converges uniformly on* $A \subset \mathbb{R}$ *to* $f : A \to \mathbb{R}$, *then* f *is continuous on* A.

Note on *This result can be interpreted as justifying an "exchange in the order of limits"*

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x).$$

Note that the uniform convergence of f_n *to* f *is sufficient, but pointwise convergence is not.*

Theorem 9.4 (Differentiability)

Suppose that (f_n) *is a sequence of differentiable functions* $f_n : (a, b) \to \mathbb{R}$ *such that* $f_n \to f$ *and* $f_n' \to g$ *uniformly for some* $f, g : (a, b) \to \mathbb{R}$. *Then* f *is differentiable on* (a, b) *and* $f' = g$.

Note on *This result can be interpreted as justifying an "exchange in the order of limits"*

$$\lim_{n\to\infty} \lim_{x\to c} \left[\frac{f_n(x) - f_n(c)}{x - c}\right] = \lim_{x\to c} \lim_{n\to\infty} \left[\frac{f_n(x) - f_n(c)}{x - c}\right].$$

9.5 Series

Definition 9.4

Suppose that (f_n) *is a sequence of functions* $f_n : A \to \mathbb{R}$. *Let* (S_n) *be the sequence of partial sums* $S_n : A \to \mathbb{R}$, *defined by*

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then the series

$$S(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges pointwise to $S : A \to \mathbb{R}$ *on* A *if* $S_n \to S$ *as* $n \to \infty$ *pointwise on* A, *and uniformly to* S *on* A *if* $S_n \to S$ *uniformly on* A.

Note on Example *The geometric series* $\sum_{n=0}^{\infty} x^n$ *has partial sums*

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Thus, $S_n(x) \to 1/(1-x)$ *as* $n \to \infty$ *if* $|x| < 1$ *and diverges if* $|x| \geq 1$, *meaning that*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{pointwise on } (-1, 1).$$

Since $1/(1-x)$ *is unbounded on* $(-1,1)$, *Theorem 9.2 implies that the convergence cannot be uniform. The series does, however, converge uniformly on* $[-\rho, \rho]$ *for every* $0 \leq \rho < 1$.

Theorem 9.5 (Cauchy condition for the uniform convergence of series)

Let (f_n) *be a sequence of functions* $f_n : A \to \mathbb{R}$. *The series* $\sum_{n=1}^{\infty} f_n$ *converges uniformly on* A *iff for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \epsilon \qquad \text{for all } x \in A \text{ and all } n > m > N.$$

Theorem 9.6 (Weierstrass M-test)

Let (f_n) *be a sequence of functions* $f_n : A \to \mathbb{R}$ *and suppose that for every* $n \in \mathbb{N}$ *there exists a constant* $M_n \geq 0$ *such that*

$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \qquad \sum_{n=1}^{\infty} M_n < \infty.$$

Then $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* A.

Note on Example *Consider the geometric series* $\sum_{n=0}^{\infty} x^n$, *if* $|x| \leq \rho$ *where* $0 \leq \rho < 1$, *then*

$$|x^n| \leq \rho^n, \qquad \sum_{n=0}^{\infty} \rho^n < 1.$$

The M-test, with $M_n = \rho^n$, *implies that the series converges uniformly on* $[-\rho, \rho]$.

Note on *If the Weierstrass M-test applies to a series of functions to prove uniform convergence,* *then it also implies that the series converges absolutely, meaning that*

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty \quad \textit{for every } x \in A.$$

Thus, the M-test is not applicable to series that converge uniformly but not absolutely. For example, let $f_n : \mathbb{R} \to \mathbb{R}$ *be the constant function* $f_n(x) = \frac{(-1)^{n+1}}{n}$*. Then* $\sum f_n$ *converges on* $\mathbb{R}$ *to the constant function* $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$*. The convergence* $\sum f_n$ *is uniform on* $\mathbb{R}$ *since the terms in the series do not depend on* x*, but the convergence isn't absolute at any* $x \in \mathbb{R}$*.*

10 Power Series

10.1 Radius of convergence

Definition 10.1

A power series (centered at 0) is a series of the form $\sum_{n=0}^{\infty} a_n x^n$ *where the constants* a_n *are some coefficients.*

Note on *Every power series has a radius of convergence* $0 \leq R \leq \infty$*, which depends on the coefficients* a_n*. The power series converges absolutely in* $|x| < R$ *and diverges in* $|x| > R$*. Moreover, the convergence is uniform on every interval* $|x| < \rho$ *where* $0 \leq \rho < R$*.*

Definition 10.2

Let $(a_n)_{n=0}^{\infty}$ *be a sequence of real numbers and* $c \in \mathbb{R}$*. The power series centered at* c *with coefficients* a_n *is the series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

Theorem 10.1

Let

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

be a power series. There is a non-negative, extended real number $0 \leq R \leq \infty$ *such that the series converges absolutely for* $0 \leq |x-c| < R$ *and diverges for* $|x-c| > R$*. Furthermore, if* $0 \leq \rho < R$*, then the power series converges uniformly on the interval* $|x-c| \leq \rho$*, and the sum of the series is continuous in* $|x-c| < R$*.*

Definition 10.3

If the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges for $|x-c| < R$ *and diverges for* $|x-c| > R$*, then* $0 \leq R \leq \infty$ *is called the radius of convergence of the power series.*

Note on *However, this theorem does not say what happens at the endpoints $x = c \pm R$, and in general the power series may converge or diverge there.*

Theorem 10.2 (Ratio test)

Suppose that $a_n \neq 0$ for all sufficiently large n and the limit

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists or diverges to infinity. Then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence R.

Theorem 10.3 (Hadamard)

The radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

where $R = 0$ if the $\limsup$ diverges to ∞, and $R = \infty$ if the $\limsup$ is 0.

Note on Examples *The geometric series $\sum_{n=0}^{\infty} x^n$ has radius of convergence*

$$R = \lim_{n\to\infty} \frac{1}{1} = 1.$$

It converges to

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \textit{for } |x| < 1,$$

and diverges for $|x| > 1$. And the interval of convergence of the power series is $(-1, 1)$.

The series $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ has radius of convergence

$$R = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

And the interval of convergence of the power series is $[-1, 1)$.

The power series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ has radius of convergence

$$R = \lim_{n\to\infty} \frac{1/n!}{1/(n+1)!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty,$$

so it converges for all $x \in \mathbb{R}$. The sum is the exponential function e^x.

The power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ has radius of convergence $R = \infty$, and it converges for all $x \in \mathbb{R}$. The sum is $\cos x$.

The power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ has radius of convergence $R = \infty$, and it converges for all $x \in \mathbb{R}$. The sum is $\sin x$.

The power series $\sum_{n=0}^{\infty} (n!) x^n$ has radius of convergence

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

so it converges only for $x = 0$.

10.2 Algebraic operations on power series

Proposition 10.1

If $R, S > 0$ *and the functions*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{in } |x| < R, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{in } |x| < S$$

are sums of convergent power series, then

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n \quad \text{in } |x| < T,$$
$$(fg)(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{in } |x| < T,$$

where $T = \min(R, S)$ *and* $c_n = \sum_{k=0}^{n} a_{n-k} b_k$.

Proposition 10.2

If $R > 0$ *and*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{in } |x| < R,$$

is the sum of a power series with $a_0 \neq 0$*, then there exists* $S > 0$ *such that*

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n x^n \quad \text{in } |x| < S.$$

The coefficients b_n *are determined recursively by*

$$b_0 = \frac{1}{a_0}, \quad b_n = -\frac{1}{a_0} \sum_{k=0}^{n-1} a_{n-k} b_k, \quad \text{for } n \geq 1.$$

Proposition 10.3

If $R, S > 0$ *and*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{in } |x| < R, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{in } |x| < S$$

are the sums of power series with $b_0 \neq 0$*, then there exists* $T > 0$ *and coefficients* c_n *such that*

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} c_n x^n \quad \text{in } |x| < T.$$

10.3 Differentiation of power series

Theorem 10.4

Suppose that the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ *has radius of convergence* R*. Then the power series*

$$\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

also has radius of convergence R.

Theorem 10.5

Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{for } |x-c| < R$$

has radius of convergence $R > 0$ and sum f. Then f is differentiable in $|x-c| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \quad \text{for } |x-c| < R.$$

Theorem 10.6

If the power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence $R > 0$, then f is infinitely differentiable in $|x-c| < R$ and

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Note on *This theorem looks similar to Taylor's theorem, Theorem 8.9, but there is a fundamental difference. Taylor's theorem gives an expression for the error between a function and its Taylor polynomials. No question of convergence is involved. On the other hand, this theorem asserts the convergence of an infinite power series to a function f. The coefficients of the Taylor polynomials and the power series are the same in both cases, but Taylor's theorem approximates f by its Taylor polynomials $P_n(x)$ of degree n at c in the limit $x \to c$ with n fixed, while the power series theorem approximates f by $P_n(x)$ in the limit $n \to \infty$ with x fixed.*

Corollary 10.1

If two power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n, \quad \sum_{n=0}^{\infty} b_n (x-c)^n$$

have nonzero-radius of convergence and are equal in some neighborhood of 0, then $a_n = b_n$ for every $n = 0, 1, 2, \ldots$.

10.4 The exponential function $E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Proposition 10.4

For every $x, y \in \mathbb{R}$,

$$E(x)E(y) = E(x+y).$$

Note on *In particular, since $E(0) = 1$, it follows that*

$$E(-x) = \frac{1}{E(x)}.$$

Proposition 10.5

Suppose that $f : \mathbb{R} \to \mathbb{R}$ *is a differentiable function such that*

$$f' = f, \quad f(0) = 1.$$

Then $f = E$.

Proposition 10.6

Suppose that n *is a non-negative integer. Then*

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0.$$

10.5 Smooth versus analytic functions

Definition 10.4

Let $K \in \mathbb{N}$. *A function* $f : (a, b) \to \mathbb{R}$ *is* C^k *on* (a, b), *written* $f \in C^k(a, b)$, *if it has continuous derivatives* $f^{(j)} : (a, b) \to \mathbb{R}$ *of orders* $1 \leq j \leq k$. *A function* f *is smooth (or* C^∞, *or infinitely differentiable) on* (a, b), *written* $f \in C^\infty(a, b)$, *if it has continuous derivatives of all orders on* (a, b).

Definition 10.5

A function $f : (a, b) \to \mathbb{R}$ *is analytic on* (a, b) *if for every* $c \in (a, b)$ *the function* f *is the sum in a neighborhood of* c *of a power series centered at* c *with nonzero radius of convergence.*

Note on *Theorem 10.6 implies that an analytic function is smooth. What is less obvious is that a smooth function need not be analytic. For example, define* $\phi : \mathbb{R} \to \mathbb{R}$ *by*

$$\phi(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Then ϕ *is smooth but not analytic on* $\mathbb{R}$. *Since* ϕ *has derivatives of all orders on* $\mathbb{R}$ *and* $\phi^{(n)}(0) = 0$ *for all* $n \geq 0$, *the Taylor series of* ϕ *at 0 therefore converges to 0, so its sum is not equal to* ϕ *in any neighborhood of 0.*

The function ϕ *illustrates that knowing the values of a smooth function and all of its derivatives at one point does not tell us anything about the values of the function at nearby points. This behavior contrasts with, and highlights, the remarkable property of analytic functions that the values of an analytic function and all of its derivatives at a single point of an interval determine the function on the whole interval.*

Definition 10.6

A function $f : \mathbb{R} \to \mathbb{R}$ *has compact support if there exists* $R \geq 0$ *such that* $f(x) = 0$ *for all* $x \in \mathbb{R}$ *with* $|x| \geq R$.

Note on *We can similarly construct C^1 or C^k functions with compact support, for example,*

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Using ϕ, however, we can construct a smooth (C^∞) function with compact support, for example, the function

$$\eta(x) = \begin{cases} \exp\left[-1/\left(1 - x^2\right)\right] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

is infinitely differentiable on $\mathbb{R}$, since $\eta(x) = \phi(1 - x^2)$ is a composition of smooth functions. Moreover, it vanishes for $|x| \geq 1$, so it is a smooth function with compact support. This function is sometimes called a 'bump' function.

Proposition 10.7

Suppose that $f, g : (a, b) \to \mathbb{R}$ are analytic functions on an open interval (a, b). If $f^{(n)}(c) = g^{(n)}(c)$ for all $n \geq 0$ at some point $c \in (a, b)$, then $f = g$ on (a, b).

Proof The proof uses a common trick of going from a local result (equality of functions in a neighborhood of a point) to a global result (equality of functions on the whole of their connected domain) by proving that an appropriate subset is open, closed, and non-empty. ■

Note on *One particular consequence of this proposition is that a non-zero analytic function on $\mathbb{R}$ cannot have compact support, since an analytic function on $\mathbb{R}$ that is equal to zero on any interval*

$$(a, b) \subset \mathbb{R}$$

must equal zero on $\mathbb{R}$.

11 The Riemann Integral

Integrability is a less restrictive condition on a function than differentiability. Generally speaking, integration makes functions smoother, while differentiation makes functions rougher. For example, the indefinite integral of every continuous function exists and is differentiable, whereas the derivative of a continuous function need not exist (and typically doesn't).

The Riemann integral is the simplest integral to define, and it allows one to integrate every continuous function as well as some not-too-badly discontinuous functions. There are, however, many other types of integrals, the most important of which is the Lebesgue integral. The Lebesgue integral allows one to integrate unbounded or highly discontinuous functions whose Riemann integrals do not exist, and it has better mathematical properties than the Riemann integral. The definition of the Lebesgue integral is more involved, requiring the use of measure theory.

11.1 The supremum and infimum of functions

Proposition 11.1

Suppose that $f, g : A \to \mathbb{R}$ *and* $f \leq g$. *Then*

$$\sup_A f \leq \sup_A g, \quad \inf_A f \leq \inf_A g.$$

Proposition 11.2

Suppose that $f : A \to \mathbb{R}$ *is a bounded function and* $c \in \mathbb{R}$. *If* $c \geq 0$, *then*

$$\sup_A cf = c \sup_A f, \quad \inf_A cf = c \inf_A f.$$

If $c < 0$, *then*

$$\sup_A cf = c \inf_A f, \quad \inf_A cf = c \sup_A f.$$

Proposition 11.3

If $f, g : A \to \mathbb{R}$ *are bounded functions, then*

$$\sup_A (f + g) \leq \sup_A f + \sup_A g, \quad \inf_A (f + g) \geq \inf_A f + \inf_A g.$$

Proposition 11.4

If $f, g : A \to \mathbb{R}$ *are bounded functions, then*

$$\left| \sup_A f - \sup_A g \right| \leq \sup_A |f - g|, \quad \left| \inf_A f - \inf_A g \right| \leq \sup_A |f - g|.$$

Proposition 11.5

If $f, g : A \to \mathbb{R}$ *are bounded functions such that*

$$|f(x) - f(y)| \leq |g(x) - g(y)| \quad \text{for all } x, y \in A,$$

then

$$\sup_A f - \inf_A f \leq \sup_A g - \inf_A g.$$

11.2 Definition of the integral

Definition 11.1

Let I *be a nonempty, compact interval. A partition of* I *is a finite collection* $\{I_1, I_2, \dots, I_n\}$ *of almost disjoint, nonempty, compact subintervals whose union is* I.

Note on *Abusing notation, we will denote a partition* P *with subintervals* $I_k = [x_{k-1}, x_k]$ *of* $[a, b]$ *either by its intervals*

$$P = \{I_1, I_2, \dots, I_n\}$$

or by the set of endpoints of the intervals

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}.$$

Definition 11.2 (Upper (Lower) Riemann sum)

If $P = \{I_1, I_2, \ldots, I_n\}$ is a partition of I, let

$$M_k = \sup_{I_k} f, \quad m_k = \inf_{I_k} f.$$

The upper and lower Riemann sum of f with respect to the partition P is defined by

$$U(f; P) = \sum_{k=1}^{n} M_k |I_k| = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$
$$L(f; P) = \sum_{k=1}^{n} m_k |I_k| = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

Definition 11.3 (Upper (Lower) Riemann integral $U(f)$ $(L(f))$)

Let $\Pi(a, b)$, or Π for short, denote the collection of all partitions of $[a, b]$. The upper and lower Riemann integral of f on $[a, b]$ is defined by

$$U(f) = \inf_{P\in\Pi} U(f; P) \quad L(f) = \sup_{P\in\Pi} L(f; P).$$

Note on *The set $\{U(f; P) : P \in \Pi\}$ of all upper Riemann sums of f is bounded from below by $m(b - a)$, and the set $\{L(f; P) : P \in \Pi\}$ of all lower Riemann sums is bounded from above by $M(b - a)$.*

Definition 11.4 (Darboux)

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if it is bounded and its upper integral $U(f)$ and lower integral $L(f)$ are equal. In that case, the Riemann integral of f on $[a, b]$, dentoed by

$$\int_a^b f(x)dx, \quad \int_a^b f, \quad \int_{[a,b]} f$$

or similar notations, is the common value of $U(f)$ and $L(f)$.

Note on *An unbounded function is not Riemann integrable. For example, define $f : [0, 1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then $\int_0^1 \frac{1}{x}dx$ isn't defined as a Riemann integral because f is unbounded. In fact, $\sup_{[0,x_1]} f = \infty$, so the upper Riemann sums of f are not well-defined.

An integral with an unbounded inteval of integration, such as $\int_1^\infty \frac{1}{x}dx$ also isn't defined as a Riemann integral.

Note on *A bounded function on a compact interval may not have the Riemann integral=. For* *example, the Dirichlet function* $f : [0,1] \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \backslash \mathbb{Q}. \end{cases}$$

And it is not Riemann integrable.

Definition 11.5

A partition $Q = \{J_1, J_2, \ldots, J_m\}$ *is a refinedment of a partition* $P = \{I_1, I_2, \ldots, I_n\}$ *if every interval* I_k *in* P *is an almost disjoint union of one or more intervals* J_ℓ *in* Q.

Note on *That is, if we represent partitions by their endpoints, then* Q *is a refinement of* P *if* $Q \supset P$*, meaning that every endpoint of* P *is an endpoint of* Q.

Theorem 11.1

Suppose that $f : [a,b] \to \mathbb{R}$ *is bounded,* P *is a partitions of* $[a,b]$*, and* Q *is refinement of* P*. Then*

$$U(f;Q) \leq U(f;P), \quad L(f;P) \leq L(f;Q).$$

Proposition 11.6

If $f : [a,b] \to \mathbb{R}$ *is bounded and* P, Q *are partitions of* $[a,b]$*, then*

$$L(f;P) \leq U(f;Q).$$

Proposition 11.7

If $f : [a,b] \to \mathbb{R}$ *is bounded, then*

$$L(f) \leq U(f).$$

11.3 The Cauchy criterion for integrability

Theorem 11.2

A bounded function $f : [a,b] \to \mathbb{R}$ *is Riemann integrable iff for every* $\epsilon > 0$ *there exists a partition* P *of* $[a,b]$*, which may depend on* ϵ*, such that*

$$U(f;P) - L(f;P) < \epsilon.$$

Note on *If* $U(f;P) - L(f;P) < \epsilon$*, then* $U(f;Q) - L(f;Q) < \epsilon$ *for every refinement* Q *of* P*, so the Cauchy condition means that a function is integrable iff its upper and lower sums get arbitrarily close together for all sufficiently refined partitions.*

Definition 11.6

The oscillation of a bounded function f *on a set* A *is*

$$\operatorname{osc}_A f = \sup_A f - \inf_A f.$$

Note on *If $f:[a,b]\to\mathbb{R}$ is bounded and $P=\{I_1,I_2,\dots,I_n\}$ is a partition of $[a,b]$, then*

$$U(f;P)-L(f;P)=\sum_{k=1}^{n}\sup_{I_k} f\cdot|I_k|-\sum_{k=1}^{n}\inf_{I_k} f\cdot|I_k|=\sum_{k=1}^{n}\operatorname{osc}_{I_k} f\cdot|I_k|\,.$$

That is, if we can find a sufficiently refined partition P such that the oscillation of f on most intervals is arbitrarily small, and the sum of the lengths of the remaining intervals (where the oscillation of f is large) is arbitrarily small, then f is Riemann integrable.

Proposition 11.8

Suppose that $f,g:[a,b]\to\mathbb{R}$ and g is integrable on $[a,b]$. If there exists a constant $C\geq 0$ such that

$$\operatorname*{osc}_I f\leq C\operatorname*{osc}_I g$$

on every integral $I\subset[a,b]$, then f is integrable.

Note on *A function is integrable if we can estimate its oscillation by the oscillation of an integrable function.*

Theorem 11.3

A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable iff there is a sequence (P_n) of partitions such that

$$\lim_{n\to\infty}\left[U(f;P_n)-L(f;P_n)\right]=0.$$

In this case,

$$\int_a^b f=\lim_{n\to\infty}U(f;P_n)=\lim_{n\to\infty}L(f;P_n)\,.$$

Note on *This theorem provides one way to prove the existence of an integral and, in some cases, evaluate it.*

11.4 Continuous and monotonic functions

Theorem 11.4

A continuous function $f:[a,b]\to\mathbb{R}$ on a compact interval is Riemann integrable.

Theorem 11.5

A monotonic function $f:[a,b]\to\mathbb{R}$ on a compact interval is Riemann integrable.

Note on *Monotonic functions needn't be continuous, and they may be discontinuous at a countably infinite number of points. For example, let $\{q_k:k\in\mathbb{N}\}$ be an enumeration of the rational numbers in $[0,1)$ and let (a_k) be a sequence of strictly positive real numbers such that $\sum_{k=1}^{\infty}a_k=1$. Define $f:[0,1]\to\mathbb{R}$ by*

$$f(x)=\sum_{k\in Q(x)}a_k,\quad Q(x)=\{k\in\mathbb{N}:q_k\in[0,x)\}\,.$$

for $x>0$, and $f(0)=0$. Note that f is increasing, by this theorem, f is Riemann integrable *on* $[0, 1]$*. However, it has a countably infinite number of jump discontinuities at every rational number in* $[0, 1)$*. Also, the function is continuous elsewhere.*

11.5 Linearity, monotonicity, and additivity

Theorem 11.6 (Linearity)

If $f : [a, b] \to \mathbb{R}$ *is integrable and* $c \in \mathbb{R}$*, then* cf *is integrable and*

$$\int_a^b cf = c \int_a^b f.$$

Theorem 11.7

If $f, g : [a, b] \to \mathbb{R}$ *are integrable functions, then* $f + g$ *is integrable, and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Theorem 11.8

If $f, g : [a, b] \to \mathbb{R}$ *are integrable functions, then* $fg : [a, b] \to \mathbb{R}$ *is integrable. If, in addition,* $g \neq 0$ *and* $1/g$ *is bounded, then* $f/g : [a, b] \to \mathbb{R}$ *is integrable.*

Theorem 11.9 (Monotonicity)

Suppose that $f, g : [a, b] \to \mathbb{R}$ *are integrable and* $f \leq g$*. Then*

$$\int_a^b f \leq \int_a^b g.$$

Theorem 11.10

Suppose that $f : [a, b] \to \mathbb{R}$ *is integrable and*

$$M = \sup_{[a,b]} f, \quad m = \inf_{[a,b]} f.$$

Then

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Theorem 11.11

If $f : [a, b] \to \mathbb{R}$ *is continuous, then there exists* $x \in [a, b]$ *such that*

$$f(x) = \frac{1}{b - a} \int_a^b f.$$

Proposition 11.9

If $f, g : [a, b] \to \mathbb{R}$ *are bounded functions and* $f \leq g$*, then*

$$U(f) \leq U(g), \quad L(f) \leq L(g).$$

Theorem 11.12

If f is integrable, then $|f|$ is integrable and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Corollary 11.1

If $f : [a, b] \to \mathbb{R}$ is integrable and $M = \sup_{[a,b]} |f|$, then

$$\left|\int_a^b f\right| \leq M(b-a).$$

Proposition 11.10

If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f \geq 0$ and $\int_a^b f = 0$, then $f = 0$.

Theorem 11.13 (Additivity)

Suppose that $f : [a, b] \to \mathbb{R}$ and $a < c < b$. Then f is Riemann integrable on $[a, b]$ iff it is Riemann integrable on $[a, c]$ and $[c, b]$. Moreover, in that case,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Definition 11.7

If $f : [a, b] \to \mathbb{R}$ is integrable, where $a < b$, and $a \leq c \leq b$, then

$$\int_b^a f = -\int_a^b f, \qquad \int_c^c f = 0.$$

11.6 Further existence results

Proposition 11.11

Suppose that $f, g : [a, b] \to \mathbb{R}$ and $f(x) = g(x)$ except at finitely many points $x \in [a, b]$. Then f is integrable iff g is integrable, and in that case

$$\int_a^b f = \int_a^b g.$$

Note on Example *For example, the function*

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

differs from the 0-function at one point. It is integrable and its integral is equal to 0.

Note on *The conclusion can fail if the functions differ at a countably infinite number of points. One reason is that we can turn a bounded function into an unbounded function by changing its* *values at an countably infinite number of points. For example, define* $f : [0,1] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} n & \text{if } x = 1/n \text{ for } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Then f *is equal to the 0-function except on the countably infinite set* $\{1/n : n \in \mathbb{N}\}$*, but* f *is unbounded and therefore it's not Riemann integrable.*

The result is still false, however, for bounded functions that differ at a countably infinite number of points. For example, the Dirichlet function is bounded and differs from the 0-function on the countably infinite set of rationals, but it isn't Riemann integrable.

Note on *The Lebesgue integral is better behaved than the Riemann intgeral in this respect: two functions that are equal almost everywhere, meaning that they differ on a set of Lebesgue measure zero, have the same Lebesgue integrals. In particular, two functions that differ on a countable set have the same Lebesgue integrals.*

Proposition 11.12

Suppose that $f : [a,b] \to \mathbb{R}$ *is bounded and integrable on* $[a,r]$ *for every* $a < r < b$*. Then* f *is integrable on* $[a,b]$ *and*

$$\int_a^b f = \lim_{r \to b^-} \int_a^r f.$$

Note on *For example, define* $f : [0,1] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} \sin(1/x) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}.$$

f *is bounded on* $[0,1]$*, is continuous and therefore integrable on* $[r,1]$ *for every* $0 < r < 1$*. It follows from this proposition that* f *is integrable on* $[0,1]$*.*

Theorem 11.14

If $f : [a,b] \to \mathbb{R}$ *is bounded function with finitely many discontinuities, then* f *is Riemann integrable.*

Note on Example *For example, define* $f : [0,2\pi] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} \sin(1/\sin x) & \text{if } x \neq 0, \pi, 2\pi \\ 0 & \text{if } x = 0, \pi, 2\pi \end{cases}.$$

Then f *is bounded and continuous except at* $x = 0, \pi, 2\pi$*, so it is integrable on* $[0,2\pi]$*.*

Define $f : [0,1/\pi] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} \operatorname{sgn}[\sin(1/x)] & \text{if } x \neq 1/n\pi \text{ for } n \in \mathbb{N}, \\ 0 & \text{if } x = 0 \text{ or } x \neq 1/n\pi \text{ for } n \in \mathbb{N}, \end{cases}.$$

Then f *oscillates between 1 and -1 a countably infinite number of times as* $x \to 0^+$*. It has jump discontinuities at* $x = 1/(n\pi)$ *and an essential discontinuity at* $x = 0$*. Nevertheless, it* *is Riemann integrable. To see this, note that f is bounded on $[0, 1]$ and piecewise continuous with finitely many discontinuities on $[r, 1]$ for every $0 < r < 1$. Theorem 11.14 implies that f is Riemann integrable on $[r, 1]$, and then Proposition 11.12 implies that f is integrable on $[0, 1]$.*

11.7 Riemann sums

Definition 11.8 (Tagged partition)

A tagged partition (P, C) of a compact interval $[a, b]$ is a partition

$$P = \{I_1, I_2, \ldots, I_n\}$$

of the interval together with a set

$$C = \{c_1, c_2, \ldots, c_n\}$$

of points such that $c_k \in I_k$ for $k = 1, ..., n$.

Note on *We think of the point c_k as a "tag" attached to the interval I_k.*

Definition 11.9 (Riemann sum)

The Riemann sum of f with respect to the tagged partition (P, C) is defined by

$$S(f; P, C) = \sum_{k=1}^{n} f(c_k) |I_k| .$$

Note on *That is, instead of using the supremum or infimum of f on the kth interval in the sum, we evaluate f at a point in the interval. Roughly speaking, a function is Riemann integrable if its Riemann sums approach the same value as the partition is refined, independently of how we choose the points $c_k \in I_k$.*

Definition 11.10 (Mesh)

The mesh (or norm) of P is the maximum length of its intervals,

$$\operatorname{mesh}(P) = \max_{1\le k\le n} |I_k| = \max_{1\le k\le n} |x_k - x_{k-1}| .$$

Definition 11.11 (Riemann)

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if there exists a number $R \in \mathbb{R}$ with the following property: For every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|S(f; P, C) - R| < \epsilon$$

for every tagged partition (P, C) of $[a, b]$ with mesh$(P) < \delta$. In that case, $R = \int_a^b f$ is the Riemann integral of f on $[a, b]$.

Theorem 11.15

A function is Riemann integrable (in the sense of Definition 11.11) iff it is Darboux integrable (in the sense of Definition 11.4). Furthermore, in that case, the Riemann and

Darboux integrals of the function are equal.

Definition 11.12 ($A_\epsilon(P)$, $B_\epsilon(P)$ **and** $S_\epsilon(P)$)

Let $f : [a, b] \to \mathbb{R}$ *be a bounded function. If* $P = \{I_1, I_2, \dots, I_n\}$ *is a partition of* $[a, b]$ *and* $\epsilon > 0$*, let* $A_\epsilon(P) \subset \{1, \dots, n\}$ *be the set of indices* k *such that*

$$\operatorname*{osc}_{I_k} f = \sup_{I_k} f - \inf_{I_k} f \geq \epsilon \quad \textit{for } k \in A_\epsilon(P).$$

Similarly, let $B_\epsilon(P) \subset \{1, \dots, n\}$ *be the set of indices such that*

$$\operatorname*{OSc}_{I_k} f < \epsilon \quad \textit{for } k \in B_\epsilon(P).$$

We denote the sum of the lengths of the intervals in P *where the oscillation of* f *is "large" by*

$$s_\epsilon(P) = \sum_{k \in A_\epsilon(P)} |I_k| \, .$$

Note on *That is, the oscillation of* f *on* I_k *is "large" if* $k \in A_\epsilon(P)$ *and "small" if* $k \in B_\epsilon(P)$.

Theorem 11.16

A function is Riemann integrable iff $s_\epsilon(P) \to 0$ *as mesh*$(P) \to 0$ *for every* $\epsilon > 0$.

Note on *Fixing* $\epsilon > 0$*, we say that* $s_\epsilon(P) \to 0$ *as mesh*$(P) \to 0$ *if for every* $\eta > 0$ *there exists* $\delta > 0$ *such that mesh*$(P) < \delta$ *implies that* $s_\epsilon(P) \to \eta$.

Note on *This theorem has the drawback that the necessary and sufficient condition for Riemann integrability is somewhat complicated and, in general, isn't easy to verify.*

11.8 The Lebesgue criterion

Definition 11.13

A set $E \subset \mathbb{R}$ *has Lebesgue measure zero if for every* $\epsilon > 0$ *there is a countable collection of open intervals* $\{(a_k, b_k) : k \in \mathbb{N}\}$ *such that*

$$E \subset \bigcup_{k=1}^{\infty} (a_k, b_k) \, , \quad \sum_{k=1}^{\infty} (b_k - a_k) < \epsilon.$$

Note on *The open intervals is this definition are not required to be disjoint, and they may "overlap."*

Note on *Every countable set* $E = \{x_k \in \mathbb{R} : k \in \mathbb{N}\}$ *has Lebesgue measure zero. To prove this, let* $\epsilon > 0$ *and for each* $k \in \mathbb{N}$ *define*

$$a_k = x_k - \frac{\epsilon}{2^{k+2}}, \quad b_k = x_k + \frac{\epsilon}{2^{k+2}}.$$

Then $E \subset \bigcup_{k=1}^{\infty} (a_k, b_k)$ *since* $x_k \in (a_k, b_k)$ *and*

$$\sum_{k=1}^{\infty} (b_k - a_k) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2} < \epsilon,$$

so the Lebesgue measure of E is equal to zero.

(The '$\epsilon/2^k$ trick used here is a common one in measure theory.)

Note on *The Dirichlet function is not Riemann integrable, however, it is Lebesgue integrable. Its Lebesgue integral is given by*

$$\int_0^1 f = 1 \cdot |A| + 0 \cdot |B|$$

where $A = [0,1] \cap \mathbb{Q}$ is the set of rational numbers in $[0,1]$, $B = [0,1] \backslash \mathbb{Q}$ is the set of irrational numbers, and $|E|$ denotes the Lebesgue measure of a set E. The Lebesgue measure of a subset of $\mathbb{R}$ is a generalization of the length of an interval which applies to more general sets. It turns out that $|A| = 0$ (countable set of real numbers) and $|B| = 1$. Thus, the Lebesgue integral of the Dirichlet function is 0.

Theorem 11.17

A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable iff it is bounded and the set of points at which it is discontinuous has Lebesgue measure zero.

Note on *In particular, every bounded function with a countable set of discontinuities is Riemann integrable, since such a set has Lebesgue measure zero. Riemann integrability of a function does not, however, imply that the function has only countably many discontinuities. For example, the Cantor set C has Lebesgue measure zero. Let $\chi_C : [0,1] \to \mathbb{R}$ be the characteristic function of the Cantor set,*

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise} \end{cases}.$$

$\chi_C(x)$ is an example of a Riemann integrable function with uncountably many discontinuities.

12 Properties and Applications of the Integral

12.1 The fundamental theorem of calculus

Theorem 12.1 (Fundamental theorem of calculus I)

If $F : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable in (a,b) with $F' = f$ where $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Note on *However, differentiable functions whose derivatives are unbounded or so discontinuous that they aren't Riemann integrable. For example, define $F : [0,1] \to \mathbb{R}$ by $F(x) = \sqrt{x}$. Then F is continuous on $[0,1]$ and differentiable in $(0,1]$, with*

$$F'(x) = \frac{1}{2\sqrt{x}} \quad \text{for } 0 < x \leq 1.$$

The function is unbounded, so F' is not Riemann integrable on $[0,1]$.

Theorem 12.2

Suppose that $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and continuous at a. Then

$$\lim_{h \to 0^+} \frac{1}{h} \int_a^{a+h} f(x)dx = f(a).$$

Theorem 12.3 (Fundamental theorem of calculus II)

Suppose that $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and $F : [a, b] \to \mathbb{R}$ is defined by

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at $a \le c \le b$, then F is differentiable at c and $F'(c) = f(c)$.

Note on *The assumption that f is continuous is needed to ensure that F is differentiable. For example, if*

$$f(x) = \begin{cases} 1 & \textit{for } x \ge 0 \\ 0 & \textit{for } x < 0 \end{cases},$$

then

$$F(x) = \int_0^x f(t)dt = \begin{cases} x & \textit{for } x \ge 0 \\ 0 & \textit{for } x < 0 \end{cases}.$$

The function F is continuous but not differentiable at $x = 0$.

Note on *The fundamental theorem of calculus states that differentiation and integration are inverse operations in an appropriately understood sense. The theorem has two parts: in one direction, it says roughly that the integral of the derivative is the original function; in the other direction, it says that the derivative of the integral is the original function.*

12.2 Consequences of the fundamental theorem

Theorem 12.4 (Integration by parts)

Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable in (a, b), and f', g' are integrable on $[a, b]$. Then

$$\int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx.$$

Note on Example *This can sometimes be used transform an integral into an integral that is easier to evaluate, but the importance of integration by parts goes far beyond its use as an integration technique. For example, for $n = 0, 1, 2, 3, \ldots$, let*

$$I_n(x) = \int_0^x t^n e^{-t}dt.$$

If $n \ge 1$, then integration by parts with $f(t) = t^n$ and $g'(t) = e^{-t}$ gives

$$I_n(x) = -x^n e^{-x} + n \int_0^x t^{n-1} e^{-t}dt = -x^n e^{-x} + nI_{n-1}(x).$$

It then follows by induction that

$$I_n(x) = n!\left[1 - e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}\right].$$

Theorem 12.5 (Change of variable)

Suppose that $g : I \to \mathbb{R}$ differentiable on an open interval I and g' is integrable on I. Let $J = g(I)$. If $f : J \to \mathbb{R}$ continuous, then for every $a, b \in I$,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Note on *One consequence of the second part of the fundamental theorem, is that every continuous function has an antiderivative, even if it can't be expressed explicitly in terms of elementary functions. This provides a way to define transcendental functions as integrals of elementary functions. For example, by making the substitution $s = xt$,*

$$\log x + \log y = \int_1^x \frac{1}{t}dt + \int_1^y \frac{1}{t}dt = \int_1^x \frac{1}{t}dt + \int_x^{xy} \frac{1}{s}ds = \int_1^{xy} \frac{1}{t}dt = \log(xy).$$

The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

is an anti-derivative on $\mathbb{R}$ of the Gaussian function

$$f(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}.$$

Note on *Discontinuous functions may or may not have an antiderivative and typically don't.*

12.3 Integrals and sequences of functions

Theorem 12.6

Suppose that $f_n : [a, b] \to \mathbb{R}$ is Riemann integrable for each $n \in \mathbb{N}$ and $f_n \to f$ uniformly on $[a, b]$ as $n \to \infty$. Then $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n.$$

Note on *Although we can integrate uniformly convergent sequences, we cannot in general differentiate them.*

Note on *The pointwise convergence of functions is never sufficient to imply convergence of their integrals. For example, for $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by*

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n \\ 0 & \text{if } x = 0 \text{ or } 1/n \le x \le 1 \end{cases}.$$

Then $f_n \to 0$ pointwise on $[0, 1]$ but $\int_0^1 f_n = 1$ for every $n \in \mathbb{N}$.

Note on *A more serious defect of the Riemann integral is that the pointwise limit of Riemann integrable functions needn't be Riemann integrable at all, even if it is bounded. For example, let*

$\{r_k : k \in \mathbb{N}\}$ *be an enumeration of the rational numbers in* $[0, 1]$ *and define* $f_n : [0, 1] \to \mathbb{R}$ *by*

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_k \text{ for some } 1 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Then each f_n *is Riemann integrable since it differs from the zero function at finitely many points. However,* $f_n \to f$ *pointwise on* $[0, 1]$ *to the Dirichlet function* f*, which is not Riemann integrable.*

This is another place where the Lebesgue integral has better properties than the Riemann integral. The pointwise (or pointwise almost everywhere) limit of Lebesgue measurable functions is Lebesgue measurable.

Theorem 12.7

Let $f_n : [a, b] \to \mathbb{R}$ *be a sequence of differentiable functions whose derivatives* $f_n' : [a, b] \to \mathbb{R}$ *are integrable on* (a, b)*. Suppose that* $f_n \to f$ *pointwise and* $f_n' \to g$ *uniformly on* (a, b) *as* $n \to \infty$*, where* $g : (a, b) \to \mathbb{R}$ *is continuous. Then* $f : (a, b) \to \mathbb{R}$ *is continuously differentiable on* (a, b) *and* $f' = g$*.*

Note on *The key assumption in this theorem is that the derivatives* f_n' *converge uniformly, not just pointwise; the result is false if we only assume pointwise convergence of the* f_n'*.*

12.4 Improper Riemann integrals

Definition 12.1

Suppose that $f : (a, b] \to \mathbb{R}$ *is integrable on* $[c, b]$ *for every* $a < c < b$*. Then the improper integral of* f *on* $[a, b]$ *is*

$$\int_a^b f = \lim_{\epsilon \to 0+} \int_{a+\epsilon}^b f.$$

Note on *The improper integral converges if this limit exists (as a finite real number), otherwise it diverges.*

Note on *More general improper integrals may be defined as finite sums of improper integrals of the previous forms. For example, if* $f : [a, b] \setminus \{c\} \to \mathbb{R}$ *is integrable on closed intervals not including* $a < c < b$*, then*

$$\int_a^b f = \lim_{\delta \to 0+} \int_a^{c-\delta} f + \lim_{\epsilon \to 0+} \int_{c+\epsilon}^b f.$$

Definition 12.2

Suppose that $f : [a, \infty) \to \mathbb{R}$ *is integrable on* $[a, r]$ *for every* $r > a$*. Then the improper integral of* f *on* $[a, \infty)$ *is*

$$\int_a^\infty f = \lim_{r \to \infty} \int_a^r f.$$

Definition 12.3

An improper integral $\int_a^b f$ is absolutely convergent if the improper integral $\int_a^b |f|$ converges, and conditionally convergent if $\int_a^b f$ converges but $\int_a^b |f|$ diverges.

Theorem 12.8

Suppose that $f, g : I \to \mathbb{R}$ are defined on some finite or infinite interval I. If $|f| \leq g$ and the improper intergral $\int_I g$ converges, then the improper intergral $\int_I f$ converges absolutely. Moreover, an absolutely convergent improper integral converges.

12.5 Principal value integrals

Definition 12.4

If $f : [a, b]\backslash\{c\} \to \mathbb{R}$ is integrable on closed intervals not including $a < c < b$, then the principal value integral of f on $[a, b]$ is

$$p.v. \int_a^b f = \lim_{\epsilon \to 0^+} \left(\int_a^{c-\epsilon} f + \int_{c+\epsilon}^b f \right).$$

Note on *Some integrals have a singularity that is too strong for them to converge as improper integrals but, due to cancelation between positive and negative parts of the integrand, they have a finite limit as a principal value integral. For example, consider $f : [-1, 1]\backslash\{0\}$ defined by $f(x) = \frac{1}{x}$. The improper integral is*

$$\int_{-1}^1 \frac{1}{x} dx = \lim_{\delta \to 0^+} \int_{-1}^{-\delta} \frac{1}{x} dx + \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x} dx$$
$$= \lim_{\delta \to 0^+} \log \delta - \lim_{\epsilon \to 0^+} \log \epsilon.$$

Neither limit exists, so the improper integral diverges. If, however, we take $\delta = \epsilon$ and combine the limits, we get a convergent principal value integral, which is defined by

$$p.v. \int_{-1}^1 \frac{1}{x} dx = \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x} dx + \int_\epsilon^1 \frac{1}{x} dx \right) = \lim_{\epsilon \to 0^+} (\log \epsilon - \log \epsilon) = 0.$$

The crucial feature of a principal value integral is that we remove a symmetric interval around a singular point, or infinity. However, consider the principal value integral

$$p.v. \int_{-1}^1 \frac{1}{x^2} dx = \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x^2} dx + \int_\epsilon^1 \frac{1}{x^2} dx \right)$$
$$= \lim_{\epsilon \to 0^+} \left(\frac{2}{\epsilon} - 2 \right) = \infty.$$

In this case, there is no cancelation and the principal value integral diverges to ∞.

12.6 The integral test for series

Theorem 12.9 (Integral test)

Suppose that $f : [1, \infty) \to \mathbb{R}$ is a positive decreasing function (i.e., $0 \leq f(x) \leq f(y)$ for $x \geq y$). Let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff the improper integral $\int_1^{\infty} f(x)dx$ converges. Furthermore, the limit

$$D = \lim_{n\to\infty} \left[\sum_{k=1}^{n} a_k - \int_1^n f(x)dx \right]$$

exists, and $0 \leq D \leq a_1$.

Note on *This theorem is also useful for divergent series, since it tells us how quickly their partial sums diverge.*

12.7 Taylor's theorem with integral remainder

Theorem 12.10 (Taylor with integral remainder)

Suppose that $f : (a, b) \to \mathbb{R}$ has $n+1$ derivatives on (a, b) and f^{n+1} is Riemann integrable on every subinterval of (a, b). Let $a < c < b$. Then for every $a < x < b$,

$$f(x) = f(c) + f'(c)(x - c) + \frac{1}{2!}f''(c)(x - c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(x - t)^n dt.$$

Note on *The integral form requires slightly stronger assumptions than the Lagrange form, since we need to assume that the derivative of order $n+1$ is integrable, but its proof is straightforward once we have the integral. Moreover, the integral form generalizes to vector-valued functions $f : (a, b) \to \mathbb{R}^n$, while the Lagrange form does not.*

13 Metric, Normed and Topological Spaces

13.1 Vector (Linear) space

Definition 13.1 (Vector / Linear spaces)

Let F be either $\mathbb{R}$ or $\mathbb{C}$. X is a vector space or linear space if there exists two operations, addition (+) and scalar multiplication, such that

(1) $x + y = y + x$ for all $x, y \in X$;

(2) $(x + y) + z = x + (y + z)$ for all $x, y, z \in X$;

(3) there exists an element $0 \in X$ such that $0 + x = x$ for all $x \in X$;

(4) for each x in X there exists an element $-x \in X$ such that $x + (-x) = 0$;

(5) $c(x + y) = cx + cy$ for all $x, y \in X$, $c \in F$;

(6) $(c+d)x = cx + dx$ *for all* $x \in X$, $c, d \in F$;

(7) $c(dx) = (cd)x$ *for all* $x \in X$, $c, d \in F$;

(8) $1x = x$ *for all* $x \in X$.

13.2 Metric spaces

Definition 13.2 (Metric)

A metric d *on a set* X *is a function* $d : X \times X \to \mathbb{R}$ *such that for all* $x, y, z \in X$*:*

(1) ***symmetry.*** $d(x, y) = d(y, x)$;

(2) ***positivity.*** $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *iff* $x = y$;

(3) ***triangle inequality.*** $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 13.3 (Metric space)

A metric space (X, d) *is a set* X *with a metric* d *defined on* X.

Definition 13.4 (Metric subspace)

Let (X, d) *be a metric space. A metric subspace* (A, d_A) *of* (X, d) *consists of a subset* $A \subset X$ *whose metric* $d_A : A \times A \to \mathbb{R}$ *is the restriction of* d *to* A*; that is,* $d_A(x, y) = d(x, y)$ *for all* $x, y \in A$.

Note on *The discrete metric on any set* X *is*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

This metric is nevertheless useful in illustrating the definitions and providing counterexamples.

The absolute-value metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *on* $\mathbb{R}$ *is defined by*

$$d(x, y) = |x - y|.$$

The natural numbers $\mathbb{N}$ *and the rational numbers* $\mathbb{Q}$ *with the absolute-value metric are metric subspaces of* $\mathbb{R}$.

The ℓ^1 *metric* $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *on* $\mathbb{R}$ *is defined by*

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

The Euclidean, or ℓ^2*, metric* $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *is defined by*

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

The ℓ^∞*, or maximum, metric* $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *is defined by*

$$d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|) \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Let $C(K)$ *denote the set of continuous functions* $f : K \to \mathbb{R}$*, where* $K \subset \mathbb{R}$ *is compact; for* *example, $K = [a, b]$ is a closed, bounded interval. If $f, g \in C(K)$ define*

$$d(f, g) = \sup_{x \in K} |f(x) - g(x)| = \|f - g\|_\infty, \quad \|f\|_\infty = \sup_{x \in K} |f(x)|.$$

The function $d : C(K) \times C(K) \to \mathbb{R}$ is well-defined, since a continuous function on a compact set is bounded, and d is a metric on $C(K)$. Two functions are close with respect to this metric if their values are close at every point $x \in K$. We refer to $||f||_\infty$ as the sup-norm of f.

Definition 13.5 (Open ball $B_r(x)$, closed ball $\overline{B}_r(x)$)

Let (X, d) be a metirc space. The open ball $B_r(x)$ of radius $r > 0$ and center $x \in X$ is the set of points whose distance from x is less than r,

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

The closed ball $\overline{B}_r(x)$ of radius $r > 0$ and center $x \in X$ as the set of points whose distance from x is less than or equal to r,

$$\bar{B}_r(x) = \{y \in X : d(x, y) \leq r\}.$$

Note on *For $\mathbb{R}^2$ with the Euclidean metric, the ball $B_r(x)$ is an open disc of radius r centered at x. For the ℓ^1-metric, the ball is a diamond of diameter $2r$, and for the ℓ^∞-metric, it is a square of side $2r$.*

Consider the space $C(K)$ of continuous functions $f : K \to \mathbb{R}$ on a compact set $K \subset \mathbb{R}$ with the sup-norm metric. The ball $B_r(f)$ consists of all continuous functions $g : K \to \mathbb{R}$ whose values are within r of the values of f at every $x \in K$.

Let X be a set with the discrete metric. Then $B_r(x) = \{x\}$ consists of a single point if $0 \leq r < 1$ and $B_r(x) = X$ is the whole space if $r \geq 1$.

Definition 13.6 (Bounded set)

Let (X, d) be a metirc space. A set $A \subset X$ is bounded if there exist $x \in X$ and $0 \leq R < \infty$ such that $d(x, y) \leq R$ for all $y \in A$, meaning that $A \subset B_R(x)$.

Note on *The triangle inequality implies that $d(y, z) < R + d(x, y)$ if $d(x, z) < R$, so*

$$B_R(x) \subset B_{R'}(y) \quad \textit{for } R' = R + d(x, y).$$

Thus, if this definition holds for some $x \in X$, then it holds for every $x \in X$.

Note on *Consider the space $C(K)$ of continuous functions $f : K \to \mathbb{R}$ on a compact set $K \subset \mathbb{R}$ with the sup-norm metric. The set $\mathcal{F} \subset C(K)$ of all continuous functions $f : K \to \mathbb{R}$ such that $|f(x)| \leq 1$ for every $x \in K$ is bounded, since $d(f, 0) = \|f\|_\infty \leq 1$ for all $f \in \mathcal{F}$. The set of constant functions $\{f : f(x) = c \text{ for all } x \in K\}$ isn't bounded, since $\|f\|_\infty = |c|$ may be arbitrarily large.*

Definition 13.7 (Diameter)

Let (X, d) be a metirc space and $A \subset X$. The diameter $0 \leq \operatorname{diam} A \leq \infty$ of A is

$$\operatorname{diam} A = \sup\{d(x, y) : x, y \in A\}.$$

Note on *It follows from the definitions that A is bounded iff $\operatorname{diam} A \leq \infty$.*

13.3 Normed spaces

Definition 13.8 (Normed vector space)

A normed vector space $(X, ||\cdot||)$ is a vector space X together with a function $||\cdot|| : X \to \mathbb{R}$, called a norm on X, such that for all $x, y \in X$ and $k \in \mathbb{R}$:

(1) $0 \leq \|x\| < \infty$ *and* $\|x\| = 0$ *iff* $x = 0$*;*

(2) $\|kx\| = |k|\|x\|$*;*

(3) $\|x + y\| \leq \|x\| + \|y\|$.

Proposition 13.1

If $(X, ||\cdot||)$ is a normed vector space, then $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on X.

Note on *A metric associated with a norm has the additional properties that for all $x, y, z \in X$ and $k \in \mathbb{R}$*

$$d(x + z, y + z) = d(x, y), \quad d(kx, ky) = |k|d(x, y)$$

which are called translation invariance and homogeneity, respectively. These properties imply that the open balls $B_r(x)$ in a normed space are rescaled, translated versions of the unit ball $B_1(0)$.

Definition 13.9 (Normed vector space)

- *For $1 \leq p < \infty$, the ℓ^p-norm $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$ is defined for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by*

  $$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}.$$

- *For $p = \infty$, the ℓ^∞-norm $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ is defined by*

  $$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p = \max(|x_1|, |x_2|, \ldots, |x_n|).$$

Proposition 13.2

Let $1 \leq p \leq \infty$. The space $\mathbb{R}^n$ with the ℓ^p-norm is a normed vector space.

Definition 13.10 (Normed vector space)

Let X be a vector space. Two norms $||\cdot||_a$, $||\cdot||_b$ on X are equivalent if there exist strictly positive constants $M \geq m > 0$ such that

$$m\|x\|_a \leq \|x\|_b \leq M\|x\|_a \quad \text{for all } x \in X.$$

Note on *Geometrically, two norms are equivalent if and only if an open ball with respect to either one of the norms contains an open ball with respect to the other.*

Theorem 13.1

Suppose that $1 \leq p < \infty$. Then, for every $x \in \mathbb{R}^n$,

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty.$$

Corollary 13.1

The ℓ^p and ℓ^q norms on $\mathbb{R}^n$ are equivalent for every $1 \leq p, q \leq \infty$.

13.4 Open and closed sets

Definition 13.11 (Open set and closed set)

Let X be a metric space. A set $G \subset X$ is open if for every $x \in G$ there exists $r > 0$ such that $B_r(x) \subset G$. A subset $F \subset X$ is closed if $F^c = X \backslash F$ is open.

Definition 13.12 (Neighborhood)

Let X be a metric space. A set $U \subset X$ is a neighborhood of $x \in X$ if $B_r(x) \subset U$ for some $r > 0$.

Note on *This definition then states that a subset of a metric space is open if and only if every point in the set has a neighborhood that is contained in the set. In particular, a set is open if and only if it is a neighborhood of every point in the set.*

Note on Examples *If d is the discrete metric on a set X, then every subset $A \subset X$ is open, since for every $x \in A$ we have $B_{1/2}(x) = \{x\} \subset A$. Every set is also closed, since its complement is open.*

If (X, d) is a metric space and $A \subset X$, then $B \subset A$ is open in the metric subspace (A, d_A) iff $B = A \cap G$ where G is an open subset of X.

Note on *Open sets with respect to one metric on a set need not be open with respect to another metric. For example, every subset of $\mathbb{R}$ with the discrete metric is open, but this is not true of $\mathbb{R}$ with the absolute-value metric.*

Proposition 13.3

Let X be a metric space. If $x \in X$ and $r > 0$, then the open ball $B_r(x)$ is open and the closed ball $\overline{B}_r(x)$ is closed.

Theorem 13.2

Let X be a metric space.

- *The empty set $\varnothing$ and the whole set X are open.*
- *An arbitrary union of open sets is open.*
- *A finite intersection of open sets is open.*

Theorem 13.3

Let X be a metric space.

- *The empty set $\varnothing$ and the whole set X are closed.*
- *An arbitrary intersection of closed sets is closed.*
- *A finite union of closed sets is closed.*

Definition 13.13

Let X be a metric space and $A \subset X$.

- *A point $x \in A$ is an interior point of A if $B_r(x) \subset A$ for some $r > 0$.*
- *A point $x \in A$ is an isolated point of A if $B_r(x) \cap A = \{x\}$ for some $r > 0$, meaning that x is the only point of A that belongs to $B_r(x)$.*
- *A point $x \in A$ is a boundary point of A if, for every $r > 0$, the ball $B_r(x)$ contains a point in A and a point not in A.*
- *A point $x \in A$ is an accumulation point of A if, for every $r > 0$, the ball $B_r(x)$ contains a point $y \in A$ such that $y \neq x$.*

Note on *This definition then states that a set is open if and only if every point is an interior point and closed if and only if every accumulation point belongs to the set.*

Definition 13.14

Let A be a subset of a metric space. The interior A° of A is the set of interior points of A. The boundary ∂A of A is the set of boundary points. The closure of A is $\overline{A} = A \cup \partial A$.

Proposition 13.4

Let X be a metric space and $A \subset X$. The interior of A is the largest open set contained in A,

$$A^\circ = \bigcup \{G \subset A : G \text{ is open in } X\},$$

the closure of A is the smallest closed set that contains A,

$$\overline{A} = \bigcap \{F \supset A : F \text{ is closed in } X\},$$

and the boundary of A is their set-theoretic difference,

$$\partial A = \overline{A} \backslash A^\circ.$$

Note on *It follows that the interior A° is open, the closure $\overline{A}$ is closed, and the boundary ∂A is closed. Furthermore, A is open if and only if $A = A^\circ$, and A is closed if and only if $A = \overline{A}$.*

Note on *Consider $\mathbb{R}$ with the absolute-value metric. If $I = (a, b)$ and $J = [a, b]$, then $I^\circ = J^\circ = (a, b)$, $\bar{I} = \bar{J} = [a, b]$, and $\partial I = \partial J = \{a, b\}$. If $A = \{1/n : n \in \mathbb{N}\}$, then $A^\circ = \varnothing$ and $\bar{A} = \partial A = A \cup \{0\}$. Thus, A is neither open nor closed. If $\mathbb{Q}$ is the set of rational numbers, then $\mathbb{Q}^\circ = \varnothing$ and $\overline{\mathbb{Q}} = \partial\mathbb{Q} = \mathbb{R}$.*

Suppose that X is a set containing at least two elements with the discrete metric. If $x \in X$, then the unit open ball is $B_1(x) = \{x\}$, and it is equal to its closure $\overline{B_r(x)} = \{x\}$. On the other hand, the closed unit ball is $\bar{B}_1(x) = X$. Thus, in general metric space, the closure of an open ball of radius $r > 0$ need not be the closed ball of radius r.

13.5 Completeness, compactness, and continuity

Definition 13.15

Let (X, d) be a metric space. A sequence (x_n) in X converges to $x \in X$, written $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \text{ implies that } d(x_n, x) < \epsilon.$$

Note on *If d is the discrete metric on a set X, then a sequence (x_n) converges in (X, d) iff it is eventually constant.*

Theorem 13.4

A subset $F \subset X$ of a metric space X is closed iff the limit of every convergent sequence in F belongs to F.

Definition 13.16 (Cauchy sequence)

Let (X, d) be a metric space. A sequence (x_n) in X is a Cauchy sequence for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n > N \text{ implies that } d(x_m, x_n) < \epsilon.$$

Note on *Every convergent sequence is Cauchy.*

Definition 13.17

A metric space is complete if every Cauchy sequence converges.

Note on *If d is the discrete metric on a set X, then (X, d) is a complete metric space since every Cauchy sequence is eventually constant.*

The space $(\mathbb{R}, |\cdot|)$ is complete, but the metric subspace $(\mathbb{Q}, |\cdot|)$ is not complete.

Proposition 13.5

A subspace (A, d_A) of a complete metric space (X, d) is complete iff A is closed in X.

Definition 13.18

A Banach space is a complete normed vector space.

Note on *For example, $\mathbb{R}$ with the absolute-value norm is a one-dimensional Banach space.*

Theorem 13.5

Let $1 \leq p \leq \infty$. The vector space $\mathbb{R}^n$ with the ℓ^p-norrm is a Banach space.

Definition 13.19

A subset $K \subset X$ of a metric space X is sequentially compact, or compact for short, if every sequence in K has a convergent subsequence whose limit belongs to K.

Note on *Compactness is an intrinsic property of a subset: $K \subset X$ is compact iff the metric subspace (K, d_K) is compact.*

Note on *There is a significant difference between compact sets in a general metric space and in $\mathbb{R}$. Every compact subset of a metric space is closed and bounded, as in $\mathbb{R}$, but it is not always true that a closed, bounded set is compact.*

First, a set must be complete, not just closed, to be compact. (A closed subset of $\mathbb{R}$ is complete because $\mathbb{R}$ is complete.) For example, consider the metric space $\mathbb{Q}$ with the absolute value norm. The set $[0, 2] \cap \mathbb{Q}$ is a closed, bounded subspace, but it is not compact.

Second, completeness and boundedness is not enough, in general, to imply compactness. Consider $\mathbb{N}$ with the discrete metric,

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}.$$

Then $\mathbb{N}$ is complete and bounded with respect to this metric. However, it is not compact since $x_n = n$ is a sequence with no convergent subsequence.

Definition 13.20

Let (X, d) be a metric space. A subset $A \subset X$ is totally bounded if for every $\epsilon > 0$ there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ of points in X such that

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i).$$

Theorem 13.6

A subset $K \subset X$ of a metric space X is sequentially compact iff it is complete and totally bounded.

Definition 13.21

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $c \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, c) < \delta \text{ implies that } d_Y(f(x), f(c)) < \epsilon.$$

Note on *A function $f : \mathbb{R} \to \mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with the Euclidean norm $||\cdot||$ and $\mathbb{R}$*

with the absolute value norm $|\cdot|$, *is continuous at* $c \in \mathbb{R}^2$ *if*

$$|x - c| < \delta \text{ implies that } \|f(x) - f(c)\| < \epsilon.$$

Explicitly, if $f(x) = (f_1(x), f_2(x))$, *where* $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, *this condition reads:* $|x - c| < \delta$ *implies that*

$$\sqrt{[f_1(x) - f_1(c)]^2 + [f_2(x) - f_2(c)]^2} < \epsilon.$$

Define $F : C([0, 1]) \to \mathbb{R}$ *by* $F(f) = f(0)$, *where* $C([0, 1])$ *is the space of continuous functions* $f : [0, 1] \to \mathbb{R}$ *equipped with the sup-norm, and* $\mathbb{R}$ *has the absolute value norm. That is,* F *evaluates a function* $f(x)$ *at* $x = 0$. *Thus,* F *is a function acting on functions, and its values are scalars; such a function, which maps functions to scalars, is called a functional. Then* F *is continuous, since* $\|f - g\|_\infty < \epsilon$ *implies that* $|f(0) - g(0)| < \epsilon$.

Theorem 13.7

Let X *and* Y *be metric spaces. A function* $f : X \to Y$ *is continuous at* $c \in X$ *iff* $f(x_n) \to f(c)$ *as* $n \to \infty$ *for every sequence* (x_n) *in* X *such that* $x_n \to c$ *as* $n \to \infty$.

Definition 13.22

Let (X, d_X) *and* (Y, d_Y) *be metric spaces. A function* $f : X \to Y$ *is uniformly continuous on* X *if for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$d_X(x, y) < \delta \text{ implies that } d_Y(f(x), f(y)) < \epsilon.$$

Theorem 13.8

A function $f : X \to Y$ *between metric spaces* X *and* Y *is continuous on* X *iff* $f^{-1}(V)$ *is open in* X *for every open set* V *in* Y.

Note on *A function on a metric space is continuous if and only if the inverse images of open sets are open.*

Theorem 13.9

Let $f : K \to Y$ *be a continuous function from a compact metric space* K *to a metric space* Y. *Then* $f(K)$ *is a compact subspace of* Y.

Note on *The continuous image of a compact set is compact.*

Theorem 13.10

If $f : K \to Y$ *is a continuous function on a compact set* K *to a metric space* Y, *then* f *is uniformly continuous.*

Note on *A continuous function on a compact set is uniformly continuous.*

13.6 Topological spaces

Definition 13.23

Let X be a set. A collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X is a topology on X if it satisfies the following conditions.

- *The empty set $\varnothing$ and the whole set X belong to $\mathcal{T}$.*
- *The union of an arbitrary collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.*
- *The intersection of a finite collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.*

Definition 13.24

A set $G \subset X$ is open with respect to $\mathcal{T}$ if $G \in \mathcal{T}$, and a set $F \subset X$ is closed with respect to $\mathcal{T}$ if $F^c \in \mathcal{T}$.

Definition 13.25

A topological space $(X, \mathcal{T})$ is a set X together with a topology $\mathcal{T}$ on X.

Note on *Every metric space with the open sets is a topological space. There are, however, topological spaces whose topology is not derived from any metric on the space.*

Let X be any set. Then $\mathcal{T} = \mathcal{P}(X)$ is a topology on X, called the discrete topology. In this topology, every set is both open and closed. This topology is the metric topology associated with the discrete metric on X.

Let X be any set. Then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X, called the trivial topology. The empty set and the whole set are both open and closed, and no other subsets of X are either open or closed. If X has at least two elements, then this topology is different from the discrete topology in the previous example, and it is not derived from a metric.

Definition 13.26

A topological space $(X, \mathcal{T})$ is Hausdorff if for every $x, y \in X$ with $x \neq y$ there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Note on *That is, a topological space is Hausdorff if distinct points have disjoint neighborhoods. In that case, we also say that the topology is Hausdorff. Nearly all topological spaces that arise in analysis are Hausdorff, including, in particular, metric spaces.*

Proposition 13.6

Every metric topology is Hausdorff.

Definition 13.27

Let X be a topological space. A set $K \subset X$ is compact if every open cover of K has a finite subcover. That is, if $\{G_i : i \in I\}$ is a collection of open sets such that

$$K \subset \bigcup_{i \in I} G_i,$$

then there is a finite subcollection $\{G_{i_1}, G_{i_2}, \ldots, G_{i_n}\}$ *such that*

$$K \subset \bigcup_{k=1}^{n} G_{i_k}.$$

Theorem 13.11

A metric space is compact if and only if it sequentially compact.

Note on *The Heine-Borel and Bolzano-Weierstrass properties are equivalent in every metric space. However, it does not hold for general topological spaces.*

Definition 13.28

Let X be a topological space. A sequence (x_n) in X converges to $x \in X$ if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for every $n > N$.

Definition 13.29

Let $f : X \to Y$ be a function between topological spaces X, Y. Then f is continuous at $x \in X$ if for every neighborhood $V \subset Y$ of $f(x)$, there exists a neighborhood $U \subset X$ of x such that $f(U) \subset V$. The function f is continuous on X if $f^{-1}(V)$ is open in X for every open set $V \subset Y$.

Note on *These definitions are equivalent to the corresponding "$\epsilon - \delta$" definitions in a metric space, but they make sense in a general topological space because they refer only to neighborhoods and open sets.*

For example, if X is a set with the discrete topology, then a sequence converges to $x \in X$ iff its terms are eventually equal to x, since $\{x\}$ is a neighborhood of x. Every function $f : X \to Y$ is continuous with respect to the discrete topology, then $f : X \to Y$ is continuous iff $f^{-1}(\{y\})$ is open in X for every $y \in Y$.

Let X be a set with the trivial topology. Then every sequence converges to every point $x \in X$, since the only neighborhood of x is X itself. As this example illustrates, non-Hausdorff topologies have the unpleasant feature that limits need not be unique, which is one reason why they rarely arise in analysis.

Definition 13.30

A topological space X is disconnected if there exist nonempty, disjoint open sets U, V such that $X = U \cup V$. A topological space is connected if it is not disconnected.

Theorem 13.12

Suppose that $f : X \to Y$ is a continuous map between topological spaces X and Y. Then $f(X)$ is compact if X is compact, and $f(X)$ is connected if X is connected.

Note on *Continuous functions map compact sets and connected sets to compact sets and connected sets, respectively.*

13.7 Function spaces

Definition 13.31

Let $K \subset \mathbb{R}$ be a compact set. The space $C(K)$ consists of the continuous functions $f : K \to \mathbb{R}$. Addition and scalar multiplication of functions is defined pointwise in the usual way: if $f, g \in C(K)$ and $k \in \mathbb{R}$, then

$$(f+g)(x) = f(x) + g(x), \quad (kf)(x) = k(f(x)).$$

Definition 13.32

The sup-norm of a function $f \in C(K)$ is defined by

$$\|f\|_\infty = \sup_{x \in K} |f(x)|.$$

Note on *Thus, the sup-norm on $C(K)$ is analogous to the ℓ^∞-norm on $\mathbb{R}^n$.*

Definition 13.33

A sequence (f_n) of functions $f_n : K \to \mathbb{R}$ converges uniformly on K to a function $f : K \to \mathbb{R}$ if

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

Definition 13.34

A sequence (f_n) of functions $f_n : K \to \mathbb{R}$ is uniformly Cauchy on K if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n > N \text{ implies that } \|f_m - f_n\|_\infty < \epsilon.$$

Note on *Moreover, like $\mathbb{R}$, the space $C(K)$ is complete.*

Theorem 13.13

The space $C(K)$ with the sup-norm $|| \cdot ||_\infty$ is a Banach space.

Definition 13.35

If $f : [a,b] \to \mathbb{R}$ is a Riemann integrable function, then the one-norm of f is

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

Theorem 13.14

The space $C([a,b])$ of continuous functions $f : [a,b] \to \mathbb{R}$ with the one-norm $|| \cdot ||_1$ is a normed space.

Note on *Although $C([a,b])$ equipped with the one-norm $|| \cdot ||_1$ is a normed space, it is not complete, and therefore it is not a Banach space.*

Note on *The ℓ^∞-norm and the ℓ^1-norm on the finite-dimensional space $\mathbb{R}^n$ are equivalent, but the sup-norm and the one-norm on $C([a,b])$ are not. For exmaple, for $n \in \mathbb{N}$, define the* *continuous function* $f_n : [0,1] \to \mathbb{R}$ *by*

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x \le 1/n \\ 0 & \text{if } 1/n < x \le 1 \end{cases}.$$

Then $\|f_n\|_\infty = 1$ *for every* $n \in \mathbb{N}$, *but*

$$\|f_n\|_1 = \int_0^{1/n} (1 - nx)dx = \left[x - \frac{1}{2}nx^2\right]_0^{1/n} = \frac{1}{2n},$$

so $\|f_n\|_1 \to 0$ *as* $n \to \infty$.

The incompleteness of $C([a,b])$ *with respect to the one-norm suggests that we use the larger space* $R([a,b])$ *of Riemann integrable functions on* $[a,b]$, *which includes some discontinuous functions. However, the space of Riemann integrable functions with the one-norm is still not complete. To get a space that is complete with respect to the one-norm, we have to use the space* $L^1([a,b])$ *of Lebesgue integrable functions on* $[a,b]$. *This is another reason for the superiority of the Lebesgue integral over the Riemann integral: it leads function spaces that are complete with respect to integral norms.*

13.8 The Minkowski inequality

Theorem 13.15 (Cauchy-Schwartz inequality)

If $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ *are points in* $\mathbb{R}^n$, *then*

$$\left|\sum_{i=1}^n x_i y_i\right| \le \left(\sum_{i=1}^n x_i^2\right)^{1/2} \left(\sum_{i=1}^n y_i^2\right)^{1/2}.$$

Theorem 13.16 (Minkowski inequality)

If $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ *are points in* $\mathbb{R}^n$, *then*

$$\left[\sum_{i=1}^n (x_i + y_i)^2\right]^{1/2} \le \left(\sum_{i=1}^n x_i^2\right)^{1/2} + \left(\sum_{i=1}^n y_i^2\right)^{1/2}.$$

Definition 13.36

If $1 < p < \infty$, *then the Holder conjugate* $1 < p' < \infty$ *of* p *is the number such that*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Note on *If* $p = 1$, *then* $p' = \infty$; *and if* $p = \infty$ *then* $p' = 1$.

Theorem 13.17 (Young's inequality)

Suppose that $1 < p < \infty$ *and* $1 < p' < \infty$ *is its Holder conjugate. If* $a, b \ge 0$ *are nonnegative real numbers, then*

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Moreover, there is equality iff $a^p = b^{p'}$.

Theorem 13.18 (Holder's inequality)

Suppose that $1 < p < \infty$ *and* $1 < p' < \infty$ *is its Holder conjugate. If* $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ *are points in* $\mathbb{R}^n$*, then*

$$\left|\sum_{i=1}^{n} x_i y_i\right| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^{p'}\right)^{1/p'}.$$

Theorem 13.19 (Minkowski's inequality)

Suppose that $1 < p < \infty$ *and* $1 < p' < \infty$ *is its Holder conjugate. If* $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ *are points in* $\mathbb{R}^n$*, then*

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

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